Restricted Limits on Natural Functions with Arithmetical Graphs

Jerzy Mycka*

Abstract

In this paper we consider the process of defining natural functions by the operation of infinite limit: $F(\bar{x}) = \lim_{y\to\infty, y\in A} f(\bar{x}, y)$ (also limes inferior and limes superior are taken into account). But two restrictions are assumed: the given natural function f has a graph belonging to some stage of an arithmetical hierarchy, the index of a limit runs only through a given arithmetical subset A of natural numbers.

We investigate the arithmetical class of the graph of the function F, where the respective classes of the graph of f and the set A are known. The corollary for the Turing degrees of F is formulated.

Keywords: Theory of computation, Infinite limits.

1 Introduction

The operation of infinte limits (including limes inferior and limes superior) is a natural operation on functions. The main field for the limit operation is in the mathematical analysis (for the real functions). But also the case of natural functions is considered in mathematics and computer science (for example the important Shoenfield's Limit Lemma [11]).

In this paper we use the limit operation as an 'ideal' component of computing systems. Some existing models of computation have a strong connection with the mathematical analysis and its tools. The best example is Shannon's General Purpose Analog Computer [10].

The General Purpose Analog Computer (GPAC) is a computer whose computation evolves in continuous time. The outputs are generated from the inputs by means of a dependence defined by a finite directed graph (not necessarily acyclic) where each node is one of the following boxes: *integrator*: a two-input, one-output unit with a setting for initial condition, if the inputs are unary functions u, v, then the output is the Riemann-Stieljes integral $\lambda t. \int_{t_0}^{t} u(x)dv(x) + a$, where a and t_0 are real constants defined by the initial settings of the integrator; *constant multiplier*: a one-input, one-output unit associated to a real number, if u is the input of a constant multiplier associated to the real number k, then the output is ku; *adder*: a two-input, one-output unit, if u and v are the inputs, then the output is $u + v$; *multiplier*: a two-input, one-output unit, if u and v are the inputs, then the output is uv; *constant function*: a zero-input, one-output unit, the value of the output is always 1.

Rubel in his papers [7, 8] extended this model by an introduction of new boxes to define Extended Analog Computer in the real realm. This model is similar to the GPAC but it allows, in addition, other types of units, e.g. units that solve boundary value problems

*Institute of Mathematics, M. Curie-Sklodowska University, pl. M. Curie-Sklodowskiej 1,20-031 Lublin, Poland, E-mail: jmycka@golem.umcs.lublin.pl

(here we allow several independent variables because Rubel does not seek any equivalence with existing models) and infinite limits. The EAC permits the operations of ordinary analysis, except the unrestricted taking of limits. To avoid generating too many functions in this model, limits have some restrictions: indices can go only through some subsets of real numbers. The new units add an extended computational power relatively to the GPAC. For example, the EAC can solve the Dirichlet problem for Laplace's equation on a disc and can generate the Γ function (it is known that the GPAC cannot solve these problems [7]). It is not known whether it exists a physical version of the EAC.

For natural functions and relations the main result for infinite limits is Shoenfield's Limit Lemma. This characterizes the functions computable from the halting problem as those functions of the form $\lim_{n\to\infty} g(\bar{x}, n)$ where g is a recursive function. This lemma is a fundamental tool for studying the degrees below $0'$. Additionaly, the functions from the class Δ^0_2 given in the Limit Lemma are considered as especially useful in modeling of learning processes (see [4]).

Let us recall some facts about infinite limits. For functions defined on metric space S we have: if $x_0 \in S$ and $O(x_0, \epsilon)$ is a neighbourhood of x_0, then we define (see [3]) $\limsup_{x\to x_0} f(x) = \lim_{\epsilon\to 0}[\sup_{x\in O(x_0,\epsilon)} f(x)]$ and $\liminf_{x\to x_0} f(x) = \lim_{\epsilon\to 0}[\inf_{x\in O(x_0,\epsilon)} f(x)]$. In the infinity we have then

$$\limsup_{y\to\infty} f(x) = \lim_{y\to\infty}[\sup_{x>y} f(x)],$$

$$\liminf_{y\to\infty} f(x) = \lim_{y\to\infty}[\inf_{x>y} f(x)].$$

Because $\lambda y.[\sup_{x>y} f(x)]$ is a nonincreasing function and $\lambda y.[\inf_{x>y} f(x)]$ is a nondecreasing function, thus
$\lim_{y\to\infty}[\sup_{x>y} f(x)] = \inf_y[\sup_{x>y} f(x)]$,
$\lim_{y\to\infty}[\inf_{x>y} f(x)] = \sup_y[\inf_{x>y} f(x)]$. If $\lim_{x\to\infty} f(x)$ exists, then
$\liminf_{x\to\infty} f(x) = \lim_{x\to\infty} f(x) = \limsup_{x\to\infty} f(x)$.

Here we focus our interest on natural functions. But in an analog way to Rubel[7] we restrict the indices of limits. Let $f : \mathbb{N}^k \times \mathbb{N} \to \mathbb{N}$ be a given function and $A \subset \mathbb{N}$ a given set. Now we can consider the functions F, F', F'' defined in the following way ($\bar{x} \in \mathbb{N}^k, y \in \mathbb{N}$):

$$F(\bar{x}) = \lim_{y\in A, y\to\infty} f(\bar{x}, y),$$

$$F'(\bar{x}) = \liminf_{y\in A, y\to\infty} f(\bar{x}, y),$$

$$F''(\bar{x}) = \limsup_{y\in A, y\to\infty} f(\bar{x}, y).$$

The index y of the limits is going through the set A - not necessarily through all natural numbers. Moreover, the set A must be arithmetical, i.e. it must belong to Σ^0_n or Π^0_n for some $n \in \mathbb{N}$. Also the relation $\{(\bar{x}, y, z) : f(\bar{x}, y) = z\}$ must be arithmetical (the function f must have an arithmetical graph). It means that a membership in the set A and in the mentioned above relation can be decided by computable procedures extended by the use of quantifiers.

This restrictions can be viewed as a way to build the constructive (in some sense 'computable') limit operation. Namely, in this version of infinite limits we would use the previously constructed functions and sets (by their characteristic functions) starting from the recursive functions and sets to define the new ones.

Let us add that models of computation with infinite limits are more powerful than standard ones. Classical undecidable problems (like the halting problem) in such models are decidable. Moreover, functions computable in these models can be useful in proof debugging. The standard objection to such extensions of computable systems is their unphysical character. However, we know that some results for Newtonian physics [14] or general relativity [5] may be used to harness devices employing some kind of infinite limits.

2 Preliminaries

In this section we recall the fundamental notions and denotations which are used in this paper.

A Turing machine can be given by the following description. It consists of an infinite tape for storing the input, output, and scratch working, and a finite set of internal states. All elements on a tape are strings. Without any loss of generality, we can choose some alphabet for these strings; the binary alphabet is a practical choice.

The machine works in steps. In one step it scans the symbol from the current position of the tape (under the head of the machine), changes this symbol according to a current state of the machine and moves the position of the tape to the left or right with a transformation of state. Some states are distinguished as final, as soon as the machine reaches one of them, it stops. Our Turing machine model obeys to the following rules (classical constraints): (a) input is finite and (b) output is finite. Turing machines defined in the above way can be used to compute natural functions (for example with the coding of arguments and results in the binary alphabet).

A Turing machine with an oracle A where A is a subset of some cartesian product $\mathbb{N}^k, k \geq 1$ (A can be treated as a relation), is such a machine which can in any step decide whether or not the current content of the tape (interpreted in a given coding as a vector of numbers) is in the set A.

For a function $g : \mathbb{N}^n \to \mathbb{N}$ its graph will be denoted as $G_g = \{(\bar{x}, y) : g(\bar{x}) = y\}$. We will use an arithmetical hierarchy to classify subsets of $\mathbb{N}$ and natural functions by their graphs. This infinite hierarchy consists of the classes $\Sigma^0_0, \Pi^0_0, \ldots, \Sigma^0_i, \Pi^0_i, \ldots$. Each class $\Sigma^0_i, \Pi^0_i, i \geq 0$ is a family of relations (including sets) on some cartesian product of the set of natural numbers. The method of a construction is inductive: the classes $\Sigma^0_0 = \Pi^0_0$ contain recursive relations (i.e. these ones with characteristic functions computable by Turing machines); the class Σ^0_{n+1} includes only such elements S for which the relation S is equivalent to some relation $\exists t P$, where P is in Π^0_n; the class Π^0_{n+1} includes only such elements for which the relation S is equivalent to some relation $\forall t P$, where P is in Σ^0_n. By the Δ^0_n we denote $\Sigma^0_n \cap \Pi^0_n$. The artithmetical hierarchy satisfies the strict inclusions of its levels: $\Sigma^0_n \subset \Pi^0_{n+1}, \Pi^0_n \subset \Sigma^0_{n+1}, n \geq 0$.

The importance of the arithmetical hierarchy is connected with many fields. It can be observed as a kind of formal description of definiability (see [9]). Its classes can be used to classify 'a complexity' of mathematical notions (e.g. the definition of a limit of sequences is of Π^0_3 class). From the point of view of computability theory we can see the arithmetical hierarchy as the levels of natural functions (given by their graphs), which are different in quantity of infinite 'while' loops necessary to their computation. Also linguistic problems of computer science can be expressed in terms of this hierarchy. The most known example is the one of the classes of recursive (Σ^0_0) and recursive enumerable (Σ^0_1) languages (compare [6]).

The other important method to classify unsolvable problems is given by the notions of a jump and a reduction. The jump $A' \subset \mathbb{N}$ of the $A \subset \mathbb{N}$ is defined by $A' = \{n : f^A_n(n) \text{defined}\}$, where f_n^A is the function computed by the n-th Turing machine with an oracle A (with respect to some given effective enumeration of all Turing machines with an oracle A). The i-th jump of A is defined inductively: $A^{(0)} = A, A^{(i+1)} = (A^{(i)})'$.

We say that a function f is Turing reducible to a function g, when f can be computed by some Turing machine with an oracle G_g. We denote by $f \leq_T g$ that f is Turing reducible to g. Because $\leq_T$ is a reflexive and transitive relation, so the relation $f \equiv_T g$ which holds iff $f \leq_T g$ and $g \leq_T f$ is an equivalence relation. The Turing degree is an equivalence class of the natural functions with respect to the relation of mutual Turing reducidablity (Turing equivalence) $\equiv_T$. These classes are partially ordered by the relation induced from $\leq_T$.

The above notions (Turing reducibility, Turing equivalence, Turing classes) can be used for sets and relations. In this meaning it is sufficient to think about Turing machines which compute characteristic functions of aprioprate sets and relations. In this context it is used the following notation: $0^{(0)}$ is the Turing degree which contains all the recursive sets with its representative $\emptyset$ and by $0^{(i)}$ the Turing degree with its representative $\emptyset^{(i)}$.

We can say that, intuitively, belonging to the higher Turing degree is connected with more complex content of a set and its elements have more difficult computation as an effect. Turing degrees are important objects in computer science. They are analysed from an algebraic point of view (a structure of degrees as a partially ordered set, its density, linearity, etc) and in a computational context (problems of undecidabilty). The concept of Turing degree yields an extension of universal computation to classical unsolvable problems and allows to classify problems by their computational (un)decidability. Also as a tool Turing degrees have the imporatant power, for example in information theory randomness of sequences can be measured by using reducibilities (see [1]).

In the next sections of the paper we will use some operations on the families of sets and relations. For this purpose we will use some extensions of standard operations on sets and relations [13]. Let us define $\neg\alpha = \{A : \mathbb{N}^k - A \in \alpha\}$, where α is a family of subsets of $\mathbb{N}^k$ ($\alpha \subset 2^{\mathbb{N}^k}$). In the similiar manner we can define $\exists\alpha = \{A' = \{\bar{x} : \exists y(\bar{x}, y) \in A\} : A \in \alpha\}$, $\alpha \vee \beta = \{A \cup B : A \in \alpha, B \in \beta\}$ where $\alpha, \beta \subset 2^{\mathbb{N}^k}$. As usual for $\alpha, \beta \subset 2^{\mathbb{N}^k}$ we define additionally $\forall\alpha = \neg\exists\neg\alpha$, $\alpha \wedge \beta = \neg(\neg\alpha \vee \neg\beta)$, $\alpha \Rightarrow \beta = \neg\alpha \vee \beta$.

3 Limes inferior and limes superior

In this section we will present results useful in further considerations. We begin with an analysis of the operations of limes inferior and limes superior on natural functions.

First we will create the function $F : \mathbb{N}^k \to \mathbb{N}$ by limes inferior which indices going through some arithmetical set A for a given function $f : \mathbb{N}^{k+1} \to \mathbb{N}$ such that its graph $G_f = \{(\bar{x}, i, y) : y = f(\bar{x}, i)\}$ is an arithmetical set. We start with a lemma which shows the influence of the operation of infimum on arithmetical relations and sets.

Lemma 3.1 *Let $R_I^{A,f}(n, \bar{x}, t)$ be a a relation in $\mathbb{N}^{k+2}$ defined as below*

$$R_I^{A,f}(n, \bar{x}, t) \iff \inf_{i \geq n, i \in A} f(\bar{x}, i) = t.$$

Then for $A \in \Sigma_m^0 \cup \Pi_m^0$ and $G_f \in \Sigma_n^0 \cup \Pi_n^0$ we have

$$R_I^{A,f} \in \Pi^0_{\max(m,n)+\rho(A,G_f)+1}$$

where

$$\rho(A,G_f) = \begin{cases} 0 & (\max(m,n) = m \wedge A \in \Sigma^0_m - \Pi^0_m) \\ & \vee(\max(m,n) = n \wedge G_f \in \Sigma^0_n - \Pi^0_n) \\ & \vee(m = n \wedge A \in \Sigma^0_m - \Pi^0_m \wedge G_f \in \Sigma^0_n - \Pi^0 n) \\ 1 & \textit{otherwise} \end{cases}$$

Proof. To simplify the proof we use the fact that the relation $R_I^{A,f}$ is satisfied for $n, \bar{x}, t$ iff t is the greatest lower bound for $f(\bar{x},0),\ldots,f(\bar{x},i),\ldots$ where all the indices i of this sequence should be in A. So if $lb^{A,f}(n,\bar{x},t)$ denotes that t is the lower bound of the function $f(\bar{x},j)$ for $j \in A, j \geq n$ then

$$R_I^{A,f}(n,\bar{x},t) \iff lb^{A,f}(n,\bar{x},t) \wedge (\forall s) lb^{A,f}(n,\bar{x},s) \Rightarrow s \leq t.$$

Precisely the relation $lb^{A,f}$ used above is defined as follows:

$$lb^{A,f}(n,\bar{x},t) \iff (\forall j \geq n, j \in A) f(\bar{x},j) \geq t.$$

Now we transform the relation $lb^{A,f}$ in the more convenient form in this case:

$$lb^{A,f}(n,\bar{x},t) \iff \forall j[(j \geq n \wedge j \in A) \Rightarrow f(\bar{x},j) \geq t]$$

$$\iff \forall j[\neg(j \geq n \wedge j \in A) \vee f(\bar{x},j) \geq t]$$

$$\iff \forall j[\neg(j \geq n \wedge j \in A) \vee \forall y(G_f(\bar{x},j,y) \Rightarrow y \geq t)].$$

Then for A treated as a relation and $\geq$ represented by the relation $R_{\geq}$ we obtain:

$$lb^{A,f}(n,\bar{x},t) \iff \forall j[\neg R_{\geq}(j,n) \vee \neg A(j) \vee \forall y(\neg G_f(\bar{x},j,y) \vee R_{\geq}(y,t))].$$

Now we want to find the class $\mathcal{X}$ of the arithmetical hierarchy to which the relation $lb^{A,f}$ belongs in the case when classes of the relations G_f and A are known: $G_f \in \Sigma^0_n \cup \Pi^0_n$, $A \in \Sigma^0_m \cup \Pi^0_m$. ¿From the above definition of $lb^{A,f}$ it is clear that:

$$\mathcal{X} \subset \forall[\neg\{R_{\geq}\} \vee \neg\mathcal{A} \vee \forall(\neg\mathcal{G} \vee \{R_{\geq}\})],$$

where $\mathcal{A}, \mathcal{G}$ are the classes of the arithmetical hierarchy which contain A and G_f respectively.

In the next part of the proof we will use well known properties of the classes of the arithmetical hierarchy to analyse the four possible cases:

Let $A \in \Sigma^0_m - \Pi^0_m$, $G_f \in \Sigma^0_n - \Pi^0_n$. Then we have:

$$lb^{A,f} \in \forall[\neg\{R_{\geq}\} \vee \neg\Sigma^0_m \vee \forall(\neg\Sigma^0_n \vee \{R_{\geq}\})] \subset \forall[\neg\Sigma^0_0 \vee \neg\Sigma^0_m \vee \forall(\neg\Sigma^0_n \vee \Sigma^0_0)]$$

$$= \forall[(\neg\Sigma^0_m) \vee \forall(\neg\Sigma^0_n)] = \forall[\Pi^0_m \vee \forall\Pi^0_n].$$

It is known that $\forall\Pi^0_i = \Pi^0_i$ for $i > 0$, so if $A \in \Sigma^0_m - \Pi^0_m$, $G_f \in \Sigma^0_n - \Pi^0_n$ are not empty then $m, n > 0$ and $\forall[\Pi^0_m \vee \forall\Pi^0_n] = \Pi^0_{\max(m,n)}$.

Let $A \in \Pi^0_m$, $G_f \in \Sigma^0_n - \Pi^0_n$. Then

$$lb^{A,f} \in \forall[\neg\{R_{\geq}\} \vee \neg\Pi^0_m \vee \forall(\neg\Sigma^0_n \vee \{R_{\geq}\})] \subset \forall[\neg\Pi^0_0 \vee \neg\Pi^0_m \vee \forall(\neg\Sigma^0_n \vee \Pi^0_0)]$$

$$= \forall[\Sigma^0_0 \vee \Sigma^0_m \vee \forall(\Pi^0_n \vee \Pi^0_0)] = \forall[\Sigma^0_m \vee \forall\Pi^0_n]$$

By analogy with the previous case we obtain $n > 0$, hence:

$$lb^{A,f} \in \forall[\Sigma^0_m \vee \Pi^0_n] \subset \begin{cases} \forall\Sigma^0_m = \Pi^0_{m+1} & m > n \\ \forall[\Pi^0_{m+1} \vee \Pi^0_n] = \Pi^0_{m+1} & m = n \\ \forall\Pi^0_n = \Pi^0_n & m < n. \end{cases}$$

Now for $A \in \Sigma^0_m - \Pi^0_m$, $G_f \in \Pi^0_n$ we have:

$$lb^{A,f} \in \forall[\neg\{R_\geq\} \vee \neg\Sigma^0_m \vee \forall(\neg\Pi^0_n \vee \{R_\geq\})] \subset \forall[\neg\Sigma^0_0 \vee \neg\Sigma^0_m \vee \forall(\neg\Pi^0_n \vee \Sigma^0_0)]$$
$$= \forall[\Pi^0_m \vee \forall\Sigma^0_n] = \forall[\Pi^0_m \vee \Pi^0_{n+1}] = \Pi^0_{\max(m,n+1)}.$$

And the last case: $A \in \Pi^0_m$, $G_f \in \Pi^0_n$.

$$lb^{A,f} \in \forall[\neg\{R_\geq\} \vee \neg\Pi^0_m \vee \forall(\neg\Pi^0_n \vee \{R_\geq\})] \subset \forall[\neg\Sigma^0_0 \vee \neg\Pi^0_m \vee \forall(\neg\Pi^0_n \vee \Sigma^0_0)]$$
$$= \forall[(\Sigma^0_m) \vee \forall(\Sigma^0_n)] = \forall[\Sigma^0_m \vee \Pi^0_{n+1}].$$

With respect to the m, n we obtain:

$$lb^{A,f} \in \forall[\Sigma^0_m \vee \Pi^0_{n+1}] \subset \begin{cases} \forall\Sigma^0_m = \Pi^0_{m+1} & m > n+1 \\ \forall[\Pi^0_{m+1} \vee \Pi^0_{n+1}] = \Pi^0_{m+1} & m = n+1 \\ \forall\Pi^0_{n+1} = \Pi^0_{n+1} & m < n+1 \end{cases}$$

and finally

$$lb^{A,f} \in \begin{cases} \Pi^0_{n+1} & n \geq m \\ \Pi^0_{m+1} & n < m. \end{cases}$$

The above results can be summarized in the following way. Let the set A be in $\Sigma^0_m \cup \Pi^0_m$ and the graph $G_f \in \Sigma^0_n \cup \Pi^0_n$. ¿From the previous considerations we can observe that the relation $lb^{A,f}$ is always in the class Π^0_k, where k is equal to $\max(m,n)$ in some cases incremented by one. Precisely: if $\rho(A, G_f)$ will be defined as 0 for the disjunction of the conditions $m > n \wedge A \in \Sigma^0_m - \Pi^0_m$, $n > m \wedge G_f \in \Sigma^0_n - \Pi^0_n$, $m = n \wedge A \in \Sigma^0_m - \Pi^0_m \wedge G_f \in \Sigma^0_n - \Pi^0_n$ and $\rho(A, G_f)$ will be defined as 1 otherwise then:

$$lb^{A,f} \in \Pi^0_{\max(m,n)+\rho(A,G_f)}.$$

Now let us recall the obvious fact that $R^{A,f}_I$ is satisfied for $n, \bar{x}, t$ iff t is the greatest lower bound for $f(\bar{x}, 0), \ldots, f(\bar{x}, i), \ldots$ e.g.

$$R^{A,f}_I(n, \bar{x}, t) \iff lb^{A,f}(n, \bar{x}, t) \wedge (\forall s) lb^{A,f}(n, \bar{x}, s) \Rightarrow s \leq t.$$

Again we use the operations on families on relations to establish the class $\mathcal{Y}$ to which $R^{A,f}_I$ belongs:

$$\mathcal{Y} = \mathcal{X} \wedge \forall[\mathcal{X} \Rightarrow \{R_\leq\}] = \mathcal{X} \wedge \forall[\neg\mathcal{X} \vee \{R_\leq\}] \subset \mathcal{X} \wedge \forall[\neg\mathcal{X} \vee \Sigma^0_0]$$

where $\mathcal{X}$ is the class of the arithmetical hierarchy which contains $lb^{A,f}$. If $\mathcal{X} = \Pi^0_t$ then

$$\mathcal{Y} \subset \Pi^0_t \wedge \forall[\neg\Pi^0_t \vee \Sigma^0_0] = \Pi^0_t \wedge \forall[\Sigma^0_t \vee \Sigma^0_0] = \Pi^0_t \wedge \forall\Sigma^0_t = \Pi^0_t \wedge \Pi^0_{t+1} = \Pi^0_{t+1}.$$

The proof is finished by substituting t by the proper expression: $\max(m,n) + \rho(A, G_f)$. $\square$

We are ready to prove the main result in this section. Here we present the connection between the initial arithmetical classes of some given function and indices and the class obtained after an application of limes inferior to this function.

Theorem 3.2 *Let $f : \mathbb{N}^{k+1} \to \mathbb{N}$ be a function such that $G_f \in \Sigma^0_n \cup \Pi^0_n$. Then for $A \in \Sigma^0_m \cup \Pi^0_m$ and*

$$F(\bar{x}) = \liminf_{y \in A, y \to \infty} f(\bar{x}, y)$$

we have

$$G_F \in \Pi^0_{\max(m,n)+\rho(A,G_f)+3}$$

where ρ is defined as in the Lemma 3.1.

Proof. Let us recall that

$$\liminf_{y\in A, y\to\infty} f(\bar{x},y) = \sup_n \inf_{i\geq n, i\in A} f(\bar{x},i) = \sup_n \{t : R_I^{A,f}(n,\bar{x},t)\}.$$

For the graph of the function F we have

$$G_F(\bar{x},y) \iff y = \sup_n \{t : R_I^{A,f}(n,\bar{x},t)\} \iff$$

$$\forall n \forall t[R_I^{A,f}(n,\bar{x},t) \Rightarrow t \leq y] \wedge \forall w[\forall n \forall t(R_I^{A,f}(n,\bar{x},t) \Rightarrow t \leq w) \Rightarrow (y \leq w)].$$

That last relation is equivalent to the one below:

$$\forall n \forall t[\neg R_I^{A,f}(n,\bar{x},t) \vee R_{\leq}(t,y)] \wedge \forall w[\exists n \exists t(R_I^{A,f}(n,\bar{x},t) \wedge \neg R_{\leq}(t,w)) \vee R_{\leq}(y,w)].$$

Because the class of G_F is determined by the class of $R_I^{A,f}$ we can - as in the previous lemma - transform our problem in the form which is connected rather with families of relations than with relations:

$$G_F \in \forall\forall[\neg\mathcal{Y} \vee \{R_{\leq}\}] \wedge \forall[\exists\exists(\mathcal{Y} \wedge \neg\{R_{\leq}\}) \vee \{R_{\leq}\}],$$

where $\mathcal{Y}$ is a class of the relation $R_I^{A,f}$ which always belongs to some Π^0_k so we have:

$$\begin{aligned}
&\forall\forall[\neg\Pi^0_k \vee \{R_{\leq}\}] \wedge \forall[\exists\exists(\Pi^0_k \wedge \neg\{R_{\leq}\}) \vee \{R_{\leq}\}] \\
&= \forall[\neg\Pi^0_k \vee \{R_{\leq}\}] \wedge \forall[\exists(\Pi^0_k \wedge \neg\{R_{\leq}\}) \vee \{R_{\leq}\}] \\
&\subset \forall[\neg\Pi^0_k \vee \Sigma^0_0] \wedge \forall[\exists(\Pi^0_k \wedge \neg\Sigma^0_0) \vee \Sigma^0_0] \\
&= \forall[\Sigma^0_k \vee \Sigma^0_0] \wedge \forall[\exists(\Pi^0_k \wedge \Pi^0_0) \vee \Sigma^0_0] \\
&= \forall[\Sigma^0_k] \wedge \forall[\exists\Pi^0_k \vee \Sigma^0_0] = \Pi^0_{k+1} \wedge \forall[\Sigma^0_{k+1} \vee \Sigma^0_0] \\
&= \Pi^0_{k+1} \wedge \forall\Sigma^0_{k+1} = \Pi^0_{k+1} \wedge \Pi^0_{k+2} = \Pi^0_{k+2}.
\end{aligned}$$

Now it is sufficient to use the result of the Lemma 3.1 to end the proof. □

The next theorem is simply a symmetric version of the previous one. Of course, by an obvious similarity of the notions supremum and infimum we get the below result for limes superior by a slight modification in the proof for limes inferior.

Theorem 3.3 *Let $f : \mathbb{N}^{k+1} \to \mathbb{N}$ be a function such that $G_f \in \Sigma^0_n \cup \Pi^0_n$. Then for $A \in \Sigma^0_m \cup \Pi^0_m$ and*

$$F(\bar{x}) = \limsup_{y\in A, y\to\infty} f(\bar{x},y)$$

we have

$$G_F \in \Pi^0_{\max(m,n)+\rho(A,G_f)+3}$$

where the function ρ is the same as in the previous theorem.

Proof. We will follow the proofs of the Theorem 3.2 and Lemma 3.1. The similar denotations for upper bound

$$ub^{A,f}(n,\bar{x},t) \iff (\forall j \geq n, j \in A) f(\bar{x},j) \leq t$$

and for the restricted least upper bound

$$R_S^{A,f}(n,\bar{x},t) \iff ub^{A,f}(n,\bar{x},t) \wedge (\forall s) ub^{A,f}(n,\bar{x},s) \Rightarrow s \geq t$$

lead us to the equation:

$$G_F(\bar{x},y) \iff y = \inf_n \{t : R_S^{A,f}(n,\bar{x},t)\} \iff$$

$$\forall n \forall t[\neg R_S^{A,f}(n,\bar{x},t) \vee R_{\geq}(t,y)] \wedge \forall w[\exists n \exists t(R_S^{A,f}(n,\bar{x},t) \wedge \neg R_{\geq}(t,w)) \vee R_{\geq}(y,w)].$$

Because the only difference with the adequate equation in the proof of the Theorem 3.2 is a replacement of $R_{\leq}$ by $R_{\geq}$ so the theorem holds. □

4 Infinite limits

Here we give the main results of the paper. First we will consider the case of infinite limits on functions $f : \mathbb{N}^{n+1} \to \mathbb{N}$, where the limit is computed on some set $A \subset \mathbb{N}$. Let us rewrite that the analogous result for unrestricted limits is the Shoenfield's Limit Lemma[11]. In this section we also get as a corollary some part of this lemma. We finish this section of the paper with the theorem summarizing the influence of infinite limits with restricted indices on arithmetical relations.

The first lemma of this section formulates some possible appreciation of an effect of the operation of infinite limits on arithmetical relations.

Lemma 4.1 *Let us define the function $F : \mathbb{N}^n \to \mathbb{N}$ in the following manner:*

$$F(\bar{x}) = \lim_{y \to \infty, y \in A} f(\bar{x}, y),$$

where $A \subset \mathbb{N}$ is in $\Sigma^0_m \cup \Pi^0_m$ and $f : \mathbb{N}^{n+1} \to \mathbb{N}$, $G_f \in \Sigma^0_n \cup \Pi^0_n$ for some $n, m \in \mathbb{N}$. Then the graph G_F of the function F:

$$G_F(\bar{x}, y) \iff F(\bar{x}) = y$$

has the property below:

$$G_F \in \Sigma^0_{\max(m,n)+\phi_1(A,G_f)+1},$$

where

$$\phi_1(A, G_f) = \begin{cases} 0 & (m > n \wedge A \in \Sigma^0_m - \Pi^0_m) \\ & \vee(m < n \wedge G_f \in \Pi^0_n) \\ & \vee(m = n \wedge A \in \Sigma^0_m - \Pi^0_m \wedge G_f \in \Pi^0_n) \\ 1 & \textit{otherwise} \end{cases}$$

Proof. Let us start with an obvious equivalence:

$$G_F(\bar{x}, y) \iff \exists t \forall m R(m, t, \bar{x}, y)$$

where

$$\begin{aligned} R(m, t, \bar{x}, y) &\iff [m > t \wedge m \in A] \Rightarrow G_f(m, \bar{x}, y) \\ &\iff \neg(m > t) \vee \neg A(m) \vee G_f(m, \bar{x}, y). \end{aligned}$$

As in the previous results we use operations on families of sets and relations for our problem:

$$R \in \{R_>\} \vee \neg\mathcal{A} \vee \mathcal{G} \subset \neg\mathcal{A} \vee \mathcal{G},$$

where $\mathcal{A}, \mathcal{G}$ are classes which contain A and G_f respectively.

We will establish the class $\mathcal{R}$ of the relation R by consideration of the four cases. As the first we take $A \in \Sigma^0_m - \Pi^0_m, G_f \in \Sigma^0_n - \Pi^0_n$. Then R is in $\Pi^0_m \vee \Sigma^0_n$. We can analyse that: for $m > n$ we have $R \in \Pi^0_m$, for $n > m$ we have $R \in \Sigma^0_n$ and for $m = n$ we have $R \in \Sigma^0_n \vee \Pi^0_n \subset \Pi^0_{n+1}$. Now for $A \in \Pi^0_m, G_f \in \Sigma^0_n - \Pi^0_n$ we have $R \in \Sigma^0_m \vee \Sigma^0_n = \Sigma^0_{\max(m,n)}$. As the third case we look at $A \in \Sigma^0_m - \Pi^0_m, G_f \in \Pi^0_n$. We can consider here only the case when $m \neq 0 \neq n$, because $m = n = 0$ is the part of the first case in this proof. Now $R \in \Pi^0_m \vee \Pi^0_n = \Pi^0_{\max(m,n)}$. And at the end: if $A \in \Pi^0_m, G_f \in \Pi^0_n$ then R is in $\Sigma^0_m \vee \Pi^0_n$ and for $m > n$ we have $R \in \Sigma^0_m$, for $n > m$ we have $R \in \Pi^0_n$ and for $m = n$ we have $R \in \Sigma^0_m \vee \Pi^0_m \subset \Pi^0_{m+1}$.

Because class of G_F is equal to $\exists\forall\mathcal{R}$ so we have:

$$G_F \in \begin{cases} \exists\forall\Pi^0_m = \Sigma^0_{m+1} & A \in \Sigma^0_m - \Pi^0_m, G_f \in \Sigma^0_n - \Pi^0_n, m > n \\ \exists\forall\Sigma^0_n = \Sigma^0_{n+2} & A \in \Sigma^0_m - \Pi^0_m, G_f \in \Sigma^0_n - \Pi^0_n, m < n \\ \exists\forall\Pi^0_{n+1} = \Sigma^0_{n+2} & A \in \Sigma^0_m - \Pi^0_m, G_f \in \Sigma^0_n - \Pi^0_n, m = n \\ \exists\forall\Sigma^0_{\max(m,n)} = \Sigma^0_{\max(m,n)+2} & A \in \Pi^0_m, G_f \in \Sigma^0_n - \Pi^0_n \\ \exists\forall\Pi^0_{\max(m,n)} = \Sigma^0_{\max(m,n)+1} & A \in \Sigma^0_m - \Pi^0_m, G_f \in \Pi^0_n \\ \exists\forall\Sigma^0_m = \Sigma^0_{m+2} & A \in \Pi^0_m, G_f \in \Pi^0_n, m > n \\ \exists\forall\Pi^0_n = \Sigma^0_{n+1} & A \in \Pi^0_m, G_f \in \Pi^0_n, n > m, \\ \exists\forall\Pi^0_{m+1} = \Sigma^0_{m+2} & A \in \Pi^0_m, G_f \in \Pi^0_n, m = n, \end{cases}$$

which can be reduced to the thesis of lemma. □

The analogous characterization of the relation G_F is given in the next lemma. We use a different formulation of limit which provides us with another appreciation of the final class of the graph of the function F.

Lemma 4.2 *Let us define the function $F : \mathbb{N}^n \to \mathbb{N}$ in the following manner:*

$$F(\bar{x}) = \lim_{y\to\infty, y\in A} f(\bar{x}, y),$$

where $A \subset \mathbb{N}$ is in $\Sigma^0_m \cup \Pi^0_m$ and $f : \mathbb{N}^{n+1} \to \mathbb{N}$, $G_f \in \Sigma^0_n \cup \Pi^0_n$ for some $n, m \in \mathbb{N}$. Then the graph G_F of the function F:

$$G_F(\bar{x}, y) \iff F(\bar{x}) = y$$

has the property below:

$$G_F \in \Pi^0_{\max(m,n)+\phi_2(A,G_f)+1},$$

where

$$\phi_2(A, G_f) = \begin{cases} 0 & (m > n \wedge A \in \Sigma^0_m - \Pi^0_m) \\ & \vee(m < n \wedge G_f \in \Sigma^0_n - \Pi^0_n) \\ & \vee(m = n \wedge A \in \Sigma^0_m - \Pi^0_m \wedge G_f \in \Sigma^0_n - \Pi^0_n) \\ 1 & \textit{otherwise} \end{cases}$$

Proof. Because the expression $\lim_{y\to\infty, y\in A} f(\bar{x}, y)$ is convergent so the relation $G_F(\bar{x}, z)$ (z is a limit of $f(\bar{x}, y)$ for $y \in A$) is equivalent to

$$G_F(\bar{x}, z) \iff \forall t \exists m S(m, t, \bar{x}, z),$$

where

$$S(m, t, \bar{x}, y) \iff m > t \wedge m \in A \wedge G_f(m, \bar{x}, y)$$

Then then following inclusion holds:

$$S \in \{R_>\} \wedge \mathcal{A} \wedge \mathcal{G} \subset \mathcal{A} \wedge \mathcal{G},$$

where $\mathcal{A}, \mathcal{G}$ are classes which contain A and G_f respectively. Let us find the class $\mathcal{S}$ of the relation S by consideration of the four cases. From the similar consideration as in the above lemma we get:

$$S \in \begin{cases} \Pi^0_m & A \in \Pi^0_m, G_f \in \Sigma^0_n - \Pi^0_n, m > n \\ \Sigma^0_n & A \in \Pi^0_m, G_f \in \Sigma^0_n - \Pi^0_n, m < n \\ \Sigma^0_{m+1} & A \in \Pi^0_m, G_f \in \Sigma^0_n - \Pi^0_n, m = n \\ \Sigma^0_{\max(m,n)} & A \in \Sigma^0_m - \Pi^0_m, G_f \in \Sigma^0_n - \Pi^0_n \\ \Pi^0_{\max(m,n)} & A \in \Pi^0_m, G_f \in \Pi^0_n \\ \Sigma^0_m & A \in \Sigma^0_m - \Pi^0_m, G_f \in \Pi^0_n, m > n \\ \Pi^0_n & A \in \Sigma^0_m - \Pi^0_m, G_f \in \Pi^0_n, n > m, \\ \Sigma^0_{n+1} & A\Sigma^0_m - \in \Pi^0_m, G_f \in \Pi^0_n, m = n. \end{cases}$$

Finally the thesis follows immediately from the above equation and from the fact that G_F is a subset of $\forall\exists\mathcal{S}$. $\Box$

We turn now to the Shoenfield's Limit Lemma. This result proves that the functions computable from the halting problem can be defined by infinite limits from recursive functions. The class Δ^0_2 (which appears in the lemma) has its own importance. It plays a considerable role in degree theory (e.g. [2]), this class has also a strong connection to learning theory (for example problems of classifiable classes in [12]).

Here we can present the part of Shoenfield's Limit Lemma as the simple corollary from the above given lemmas.

Corollary 4.3 *If the function $f : \mathbb{N}^2 \to \{0,1\}$ is recursive then the set C such that $x \in C \iff \lim_{y\to\infty} f(x,y) = 1$ belongs to Δ^0_2.*

Proof. ¿From the fact that f is recursive we have $G_f \in \Sigma^0_0 = \Pi^0_0$. In this case A is equal to $\mathbb{N} \in \Pi^0_0 = \Sigma^0_0$. Let $F(x) = \lim_{y\to\infty} f(x,y)$. The first lemma in this section gives us $G_F \in \Pi^0_2$, the second one $G_F \in \Sigma^0_2$, so $G_F \in \Delta^0_2$. The characteristic function c_C of the set C can be defined by means of the characteristic function of G_F: $c_C(x) = c_{G_F}(x,1)$, so C is recursive in G_F and hence it is in Δ^0_2 class. $\Box$

By a simple comparison of the above two lemmas we can formule the theorem simplifying the resulting class of the relation G_F. In this result we have a characteristics of a degree in which the function loses its computability when it is transformed by the operation of infinite limits with indices which are in some arithmetical set.

Theorem 4.4 *If $F(\bar{x}) = \lim_{y\to\infty, y\in A} f(\bar{x},y)$, where $A \subset \mathbb{N}$ is in $\Sigma^0_m \cup \Pi^0_m$ and $f : \mathbb{N}^{n+1} \to \mathbb{N}$, $G_f \in \Sigma^0_n \cup \Pi^0_n$ for some $n,m \in \mathbb{N}$, then the graph G_F of the function F belongs to $\Delta^0_{\max(m,n)+1}$ for $m > n$ and $A \in \Sigma^0_m - \Pi^0_m$ otherwise G_F belongs to $\Delta^0_{\max(m,n)+2}$.*

5 Infinite limits and Turing degree

Let us use the above results to establish the problem of a proper Turing degree for a function (relation) which is obtained by a restricted limit. Precisely, let

$$F(\bar{x}) = \lim_{y\to\infty, y\in A} f(\bar{x},y),$$

where $A \subset \mathbb{N}$ is in $0^{(m)}$ and $f : \mathbb{N}^{n+1} \to \mathbb{N}$, $G_f \in 0^{(n)}$ for some $n,m \in \mathbb{N}$. Our question is: in which Turing degree $0^{(k)}$ is the relation G_F?

This question can be understood as a problem of finding degree of unsolvability for a new function created with two parameters: with unsolvable in some way indices of limit and with an unsolvable given function.

We simplify this problem by the condition that A, G_f are the sets $\emptyset^{(m)}$ and $\emptyset^{(n)}$, which are representatives of $0^{(m)}$ and $0^{(n)}$, respectively.

Let us start with some results concerning relations between Σ^0_n, Π^0_n sets and Turing degrees $0^{(n)}$ and sets $\emptyset^{(n)}$. The first helpful lemma is cited after Shoenfield [11].

Lemma 5.1 *The highest degree of a Σ^0_n or a Π^0_n set is $0^{(n)}$. In other words:*

$$A \in \Sigma^0_n \cup \Pi^0_n \Rightarrow A \in 0^{(k)} \wedge k \leq n.$$

Moreover: every Σ^0_n-complete or Π^0_n-complete set has the degree $0^{(n)}$.

The next result is from Odifreddi [9]

Lemma 5.2 *The set* $\emptyset^{(n)}$ *is* Σ^0_n*-complete.*

From the above results and from the previous sections we have the consequence:

Lemma 5.3 *If*

$$F(\bar{x}) = \lim_{y\to\infty, y\in A} f(\bar{x}, y),$$

and $A = \emptyset^{(m)}, G_f = \emptyset^{(n)}$ *then* $G_F \in 0^{\max(m,n)+2}$.

Proof. From $A = \emptyset^{(m)}, G_f = \emptyset^{(n)}$ we have A is Σ^0_m-complete, G_f is Σ^0_n-complete so $A \in \Sigma^0_m, G_f \in \Sigma^0_n$. In the 'worst' case from Lemma 4.1 we have $G_F \in \Sigma^0_{\max(m,n)+2}$, from Lemma 4.2 we have $G_F \in \Pi^0_{\max(m,n)+2}$, hence $G_F \in 0^{\max(m,n)+2}$. □

We can interpret this result in such a way that two kinds of unsolvability in restricted infinite limit are parallel, they do not accumulate themselves but only the greater of them is taken into account (with addition of the constant factor 2).

6 Conclusions

In this paper we introduced the method of defining natural functions by restricted limits. In this case indices of limits are taken from a given arithmetical set. Such method can be viewed as a relativization of definitions with infinite limits [11]. We found the place of functions defined in the above mentioned way in the arithmetical hierarchy. Strictly speaking we analysed graphs of such functions with respect to the arithmetical levels of functions and sets of indices given as arguments.

The usefulness of such idea can be derived from a possibility of wider applications of restricted limits in definitions of new functions. Let us observe that particular levels of the arithmetical hierarchy can be constructed by restricted limits applied to recursive functions but with indices going through properly taken sets. These sets of indices may be introduced in the same way as a result of starting from recursive sets.

We presented that the above described method of definition, in its essence conditioned by two parameters: a class of indices and a class of functions graph, in reality is connected only with the greater one (with the additional small constant). This can be interpreted as, in some sense, parallel character of computability of factors used in definitions with restricted limits. The same result is applicable to Turing degrees.

Restricted limits, as a relativization of infinite limits, can be adapted to known problems of computability theory. For example, with our results we gave an easy proof of an essential part of the Shoenfield's Limit Lemma.

As the directions of the future work we can point out the following problems:

- how such kind of restricted limits can be adequately used to model and analyse the behaviour of nonstandard Turing machines (e.g. accelerated Turing machines);

- in which way definitions with restricted limits can be useful in modeling of learning processes and information theory;

- when strengthened restrictions placed on indices will be sufficiently strong to be invariant with respect to classical computability (recursiveness)?

References

[1] R. Downey, D. Hirschfeldt, and G. LaForte, Randomness and reducibility, in *Proceeding of Mathematical Foundation of Computer Science*, LNCS 2136, 316-327, Springer-Verlag, 2001.

[2] R. Downey, G. LaForte, S. Lempp, A Δ_2^0 set with barely Σ_2^0 degree, *Journal of Symbolic Logic*, 64: 1700-1718, 1999.

[3] *Encyclopaedia of Mathematics, Upper and lower bounds*, Kluwer Academic Publishers, 1993.

[4] S. Hayashi and M. Nakata, Towards Limit Computable Mathematics, in *Types for Proofs and Programs, International Workshop, TYPES 2000*, LNCS 2277, 125-144, Springer-Verlag, 2000.

[5] G. Etesi and I. Nemeti, Non-Turing computations via Malament-Hogarth space-times, *Int.J.Theor.Phys.* 41:341-370, 2002.

[6] J. Hopcroft and J. Ullman, *Introduction to automata theory, languages and computation*, Addison-Wesely, 1979.

[7] L.A. Rubel, Some Mathematical Limitations of the General-Purpose Analog Computer, *Advances in Applied Mathematics*, 9: 22-34, 1988.

[8] L.A. Rubel, The Extended Analog Computer, *Advances in Applied Mathematics*, 14: 39-51, 1993.

[9] P. Odifreddi, *Classical Recursion Theory*, North Holland, 1989.

[10] C. Shannon, Mathematical theory of the differential analyzer, *J. Math. Phys. MIT*, 20:337-354, 1941.

[11] J.R. Shoenfield, On degrees of unsolvability, *Ann. Math.*, 69: 644-653, 1959.

[12] F. Stephan, On one-sided versus two-sided classifiaction, *Forschungsberichte Mathematische Logik*, 25/1995, Mathematische Institut, Unvesität Heidelberg, 1996.

[13] K. Weihrauch, *Computability*, Springer Verlag, 1987.

[14] Z. Xia. The existence of noncollision singularities in Newtonian systems. *The Annals of Mathematics*, 135(3):411-468, 1992.