Probabilistic Estimation of Uncertain Temporal Relations

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Abstract

A wide range of AI applications should manage time varying information, for example, temporal databases, reservation systems, keeping medical records, financial applications, planning. Many published research articles in the area of temporal representation and reasoning assume that temporal data is precise and certain, even though in reality this assumption is often false. In many situations there is a need to know the relation between two temporal intervals, as it is, for example, during query processing. Indeterminacy means that we do not know exactly when a particular event happened. When two temporal intervals are indeterminate it is in many cases impossible to derive a certain temporal relation between them.

In this paper we propose an approach to represent and estimate uncertain temporal relations by calculating the probabilities of the basic relations that can hold between two temporal primitives. We represent the relation between two temporal intervals as a matrix, four elements of which are the relations between the endpoints of these intervals. The uncertain relation between two temporal points is represented by a vector with three probability values denoting the probabilities of the basic relations (before, at the same time, after) between these points. The probabilities of Allen's interval relations between two temporal intervals are composed as joint conditional probabilities of the correspondent relations between the endpoints of the intervals. We also consider an example of using the proposed estimation mechanism, which helps to figure out possible application areas of the formalism.

Keywords: *Uncertain temporal relation, point, interval, probability.*

1 Introduction

In a wide range of AI research fields there is a need to represent and reason about temporal information. Temporal formalisms are applied, for example, in natural language understanding, planning, process control, temporal databases, i.e. in the areas, where the time course of events plays an important role. Even though temporal representation and reasoning have already achieved significant results to some extent, there still exist topics which require and deserve further research attention.

Many research articles in the area of temporal representation and reasoning assume that precise and certain temporal information is available. Generally, the proposed approaches give little or no support for situations in which imperfect temporal information exists. However, in many real applications temporal information is imperfect and we need to find some way of handling it.

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Imperfect information, according to Parsons [18], can be of five types: uncertainty, inconsistency, incompleteness, imprecision, and ignorance. In the area of temporal databases the term “indeterminacy” [13], [10] is commonly used regarding one kind of imperfect information, which is similar to imprecision from Parson’s classification. Temporal indeterminacy means that we do not know exactly when a particular event happened. Indeterminacy can arise from different sources such, that were suggested in [10]: 1) granularity mismatch (when an event is known in one granularity, and is recorded in a system with a finer granularity); 2) dating techniques (some dating techniques are inherently imprecise); 3) uncertainty in planning (projected completion dates are often inexactly specified); 4) clock measurements (every clock measurement has some imprecision).

Temporal points and intervals are the main ontological primitives proposed to be used in different temporal formalisms. Some approaches use both of them together, for example, [27], [2], and [11]. Approaches based on points or intervals as the temporal primitives have their own advantages and pitfalls. For example, Hirsch [12] pointed out, that often formalisms based on points lack the expressive power required to describe the situation adequately. At the same time, interval-based representations are more complicated compared to point-based representations. There is no a universal recommendation concerning the selection of temporal primitives, and deciding on them depends in most cases on a particular application.

In many situations there is a need to know the relation between two temporal intervals, which can be represented in two main ways: using Allen’s interval relations [1], and using four relations between the endpoints of these intervals. When the intervals are indeterminate, it is in many cases impossible to derive a certain relation between them, and the estimation of possible temporal relations can be helpful.

In this paper we propose an approach to represent and estimate uncertain relations between indeterminate temporal intervals. The uncertain relation between two intervals is represented as a matrix, four elements of which are the relations between the endpoints of these intervals. The uncertain relation is estimated by calculating the probabilities of Allen’s interval relations, that are composed as joint conditional probabilities of the relations between the endpoints of the intervals. The uncertain relation between two temporal points is represented by a vector with three probability values denoting the probabilities of the three basic relations (before, at the same time, and after) between these points.

The research area of the paper concerns temporal representation and reasoning field and the field of numerical techniques for handling imperfect information. The main issues and approaches to temporal representation and reasoning are overviewed in the survey by Chittaro and Montanari [5], and in the survey by Vila [26]. One important application area of temporal formalisms is the area of temporal databases, which is used as a background for the example in Section 6 of the paper. An overview of temporal data management in this area is made by Jenson and Snodgrass [14]. Böhlen et al. [4] discuss the main aspects of the point and the interval-based temporal data models which are used in our paper. These two temporal data models were used by Hirsch [12] in construction of the relational algebra of intervals. Hirsch also used the matrix representation of the relation between two intervals, including four relations between the endpoints, similarly to ourselves in this paper.

Many published research articles deal with imperfect information. Various approaches to this problem are mentioned in the bibliography on uncertainty management by Dyreson [8], in the surveys by Parsons [18], by Parsons and Hunter [19], by Kwan et al. [15], and by Motro [16], [17], although not many of them consider temporal imperfection. Formalisms intended for dealing with imperfection are often distinguished as symbolic and numerical. Among the numerical approaches the most well known are: probability theory, Dempster-Shafer’s theory of evidence [21], possibility theory [7], and certainty factors [22].

Van Beek [24] and van Beek and Cohen [25] discussed the representation and reasoning about temporal relations, and introduced the notion of indefinite temporal relation, which is a disjunction of the basic relations. However, no numerical measures were included in that representation compared to the representation proposed in our paper. Moreover, temporal points were supposed to be determinate only, which does not often happen in practical applications.

Dyreson and Snodgrass [9] proposed a mechanism supporting valid-time indeterminacy in temporal databases, which can be seen as an extension of the Probabilistic Data Model [3]. They represent indeterminate temporal points similarly to ourselves, although they do not consider uncertain relations between temporal intervals, and their main stress was on the development of a query language.

The probabilistic representation of uncertain relations between temporal points, which is used in this paper, was proposed by Ryabov et al. [20], together with an algebra for reasoning about uncertain temporal relations. The algebra includes negation, composition, and addition operations, which make it possible to derive unknown temporal relations in a relational network using already known uncertain relations.

The structure of this paper is the following. In the next section we present the main concepts used throughout the paper, propose the representation of uncertain relations between indeterminate temporal points and intervals. In Section 3 we propose one way to estimate the uncertain relation between two indeterminate temporal points. In Section 4 we consider the relations between the endpoints of two indeterminate temporal intervals, and distinguish the cases when the temporal values of these relations are dependent. These cases are further used in Section 5 to compose the probabilities of Allen's interval relations as joint conditional probabilities of the corresponding relations between the endpoints. In Section 6 we consider an example of using the proposed estimation mechanism. In Section 7 we address the major limitations of the formalism, and in Section 8 we make conclusions and point out some directions for further research.

2 Representation of Uncertain Temporal Relations

In this section we define the main concepts used throughout the paper including the representation of uncertain relations between indeterminate temporal points and intervals.

The various models of time that have been proposed in the literature are often classified as discrete, dense, and continuous models. We use the discrete model, which is common in the temporal databases research field. *Temporal points* are isomorphic to natural numbers, i.e. there is the notion that every point has a unique successor. The time between two points is known as a *temporal interval*. A *chronon* is an indivisible time interval of some fixed duration. A time line is represented by a sequence of chronons of identical duration. We do not specify the particular chronon size, but let it vary depending on the application. A temporal point is determinate when it is exactly known during which particular chronon it is located. Often it is not known exactly, but an interval of chronons called an *interval of indeterminacy*, during which this point can be found, is given [9].

Definition 2.1 An *indeterminate temporal point* $\mathbf{a}$ is a temporal point such that $\mathbf{a}\in[\mathbf{a}^l,\mathbf{a}^u]$, where $\mathbf{a}^l$ (lower boundary) is the first chronon of the interval of indeterminacy, $\mathbf{a}^u$ (upper boundary) is the last chronon, and $\mathbf{a}^l \leq \mathbf{a}^u$.

For many applications, it turns out that not all the chronons inside the interval of indeterminacy may be equally probable. Therefore, it is reasonable to take into account the probabilities of these chronons by defining the *probability mass function* (p.m.f.) **f(a)**. We suppose that an indeterminate temporal point $\mathbf{a}$ is attached with p.m.f. $\mathbf{f}(\mathbf{a})$ so, that $f(\mathbf{a})=0$ when $\mathbf{a}<\mathbf{a}^l$ or $\mathbf{a}>\mathbf{a}^u$; $f(\mathbf{a})\in[0,1]$ and $\sum_{\mathbf{a}=\mathbf{a}^l}^{\mathbf{a}^u}\mathbf{f}(\mathbf{a})=1$ when $\mathbf{a}\in[\mathbf{a}^l,\mathbf{a}^u]$, $f(\mathbf{a}^l)>0$, and $f(\mathbf{a}^u)>0$. The requirement that the sum of the probabilities of the chronons within the interval $[\mathbf{a}^l,\mathbf{a}^u]$ is equal to 1 results from the definition of our time ontology, according to which, a temporal point is taking place exactly during one particular chronon.

We assume that the p.m.f. is given when an indeterminate point is created. Generally, the p.m.f. stems from the sources of indeterminacy, such as granularity mismatch, dating and measurement techniques, etc. When the granularity mismatch is the source of indeterminacy, a uniform distribution (all chronons within the interval of indeterminacy are equally probable) is a useful assumption. So, if an event is known in the granularity of one hour, then in a system with the granularity of one second it is indeterminate, and we have no reason to favor one second over another if we have no any additional information. Some measurement techniques or instruments can have fixed trends in measurements, for example, the normal distribution of a variable. Figure 1 presents two examples of p.m.f.s: a "discretized" normal distribution (Figure 1a) and a uniform distribution (Figure 1b).

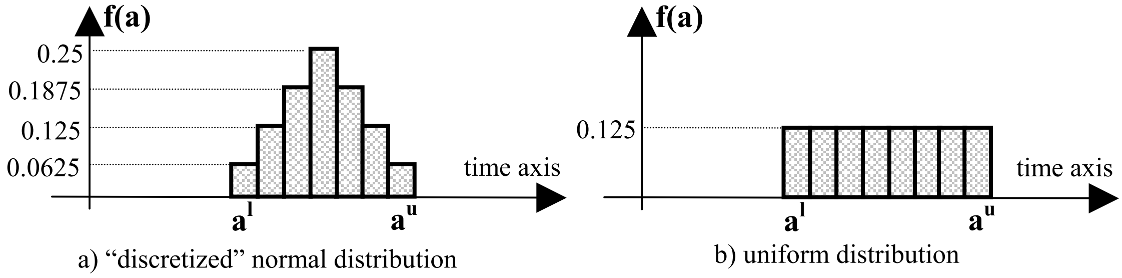

Figure 1: Examples of p.m.f.s

In some situations, the analysis of past data can provide a hint to defining the p.m.f. Several other means of determining the p.m.f. were suggested by Dey and Sarkar [6]. Also Dyreson and Snodgrass [10] pointed out that in some cases a user may not know the underlying mass function because that information is unavailable. In such cases the distribution can be specified as missing, which represents a complete lack of knowledge about the distribution. In our approach we suppose that the distribution is already and totally known. In the case when the distribution is not specified, one of the above mentioned means of defining the p.m.f. can be applied.

Definition 2.2 Let an *uncertain relation* between two temporal points **a** and **b** be represented by a vector $(\mathbf{e}^<, \mathbf{e}^=, \mathbf{e}^>)$, where the value $\mathbf{e}^<$ is the probability that $\mathbf{a}<\mathbf{b}$, the value $\mathbf{e}^=$ is the probability that $\mathbf{a}=\mathbf{b}$, and the value $\mathbf{e}^>$ is the probability that $\mathbf{a}>\mathbf{b}$. The sum of $\mathbf{e}^<$, $\mathbf{e}^=$, and $\mathbf{e}^>$ is equal to 1, since these values represent all the possible basic relations between points **a** and **b**.

Definition 2.3 Let an *indeterminate temporal interval* **A** be defined as a pair of indeterminate temporal points **s** and **e**, specifying the start and the end of the interval **A**. The starting point **s** from the interval of indeterminacy $[\mathbf{s}^l, \mathbf{s}^u]$ should be before the end point **e**, which belongs to the interval of indeterminacy $[\mathbf{e}^l, \mathbf{e}^u]$, so that the endpoints **s** and **e** do not overlap, i.e. $\mathbf{s}^u < \mathbf{e}^l$.

The relation between two temporal intervals can be represented in two main ways. The first approach is to use thirteen Allen's relations [1]: "equals" (eq), "before" (b), "after" (bi), "starts" (s), "started-by" (si), "ends" (e), "ended-by" (ei), "during" (d), "includes" (di), "overlaps" (o), "overlapped-by" (oi), "meets" (m), and "met-by" (mi). The second approach is to use four relations between the endpoints of the intervals (Figure 2).

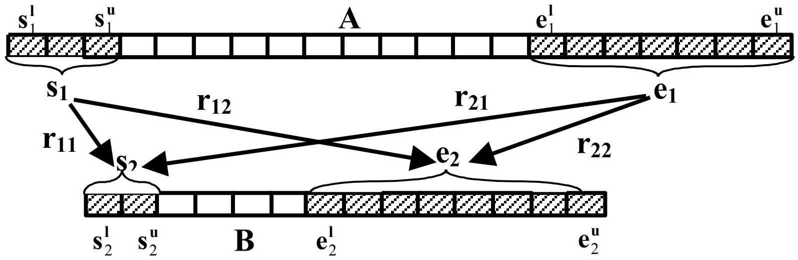

Figure 2: Relations between the endpoints of the intervals **A** and **B**

The endpoints $\mathbf{s_1}$, $\mathbf{e_1}$, $\mathbf{s_2}$, and $\mathbf{e_2}$ at Figure 2 are defined within the intervals of indeterminacy $\left[\mathbf{s}_1^{\mathbf{l}},\mathbf{s}_1^{\mathbf{u}}\right]$, $\left[\mathbf{e}_1^{\mathbf{l}},\mathbf{e}_1^{\mathbf{u}}\right]$, $\left[\mathbf{s}_2^{\mathbf{l}},\mathbf{s}_2^{\mathbf{u}}\right]$ and $\left[\mathbf{e}_2^{\mathbf{l}},\mathbf{e}_2^{\mathbf{u}}\right]$ correspondingly. The four relations between them are denoted as $\mathbf{r_{11}}$, $\mathbf{r_{12}}$, $\mathbf{r_{21}}$, and $\mathbf{r_{22}}$, and can take the values "<", "=", or ">". It is convenient to represent the relation between **A** and **B** by a matrix $\mathfrak{R} = \begin{bmatrix} \mathbf{r}_{11} & \mathbf{r}_{12} \\ \mathbf{r}_{21} & \mathbf{r}_{22} \end{bmatrix}_{\mathbf{A},\mathbf{B}}$. The uncertain relation between two indeterminate temporal intervals **A** and **B** is represented by the matrix $\mathfrak{R}$, where the relations between the endpoints are represented as in Definition 2:

$$\mathfrak{R}=\begin{bmatrix} \left(\mathbf{e}^{<},\mathbf{e}^{=},\mathbf{e}^{>}\right)_{11} & \left(\mathbf{e}^{<},\mathbf{e}^{=},\mathbf{e}^{>}\right)_{12} \\ \left(\mathbf{e}^{<},\mathbf{e}^{=},\mathbf{e}^{>}\right)_{21} & \left(\mathbf{e}^{<},\mathbf{e}^{=},\mathbf{e}^{>}\right)_{22} \end{bmatrix}_{\mathbf{A},\mathbf{B}} . \tag{1}$$

In the next section we propose one way to obtain the probability values $\mathbf{e}^{<}$, $\mathbf{e}^{=}$, and $\mathbf{e}^{>}$ for the vectors in the matrix $\mathfrak{R}$.

3 Uncertain Relation between Two Temporal Points

In this section we propose one way to estimate the uncertain temporal relation between two indeterminate temporal points by calculating the probabilities of the three basic relations that can hold between these two points. We present the formulas for these probabilities as well as the algorithmic notation. The discussion is based around an indeterminate point **a** defined within the closed interval $[\mathbf{a^l},\mathbf{a^u}]$ together with its p.m.f. $\mathbf{f_1}(\mathbf{a})$ and an indeterminate point **b** defined within the closed interval $[\mathbf{b^l},\mathbf{b^u}]$ together with the p.m.f. $\mathbf{f_2}(\mathbf{b})$.

Let us consider the composition of the probability $\mathbf{e}^{<}$ of the temporal relation "before" between points **a** and **b** using the example from Figure 3, where $\mathbf{a}\in[1,5]$ and $\mathbf{b}\in[3,9]$.

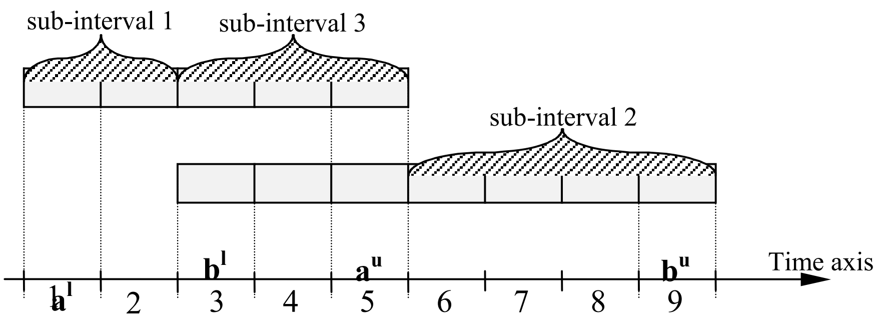

Figure 3: Division of the intervals of indeterminacy during the estimation of the relation "before"

One straightforward solution is to compare all possible pairs of the values of **a** and **b**, and in the case when $\mathbf{a}<\mathbf{b}$ the probability of the pair $\mathbf{f_1}(\mathbf{a})\times\mathbf{f_2}(\mathbf{b})$ contributes to the probability $\mathbf{e}^{<}$. This solution is very expensive from the computational point of view in the case when the intervals of indeterminacy contain many chronons. At the same time, we suggest that there is no need to compare all possible pairs of the values of **a** and **b**, but only those ones that are common to both of the intervals of indeterminacy. Therefore, we divide the intervals $[\mathbf{a^l},\mathbf{a^u}]$ and $[\mathbf{b^l},\mathbf{b^u}]$ into subintervals by filled braces as it is shown in Figure 3.

Sub-interval 1 defined as $[\mathbf{a^l},\mathbf{b^l}\text{-}1]$ contains those chronons from the interval $[\mathbf{a^l},\mathbf{a^u}]$ that are not included in the interval $[\mathbf{b^l},\mathbf{b^u}]$. Sub-interval 2 defined as $[\mathbf{a^u}+1,\mathbf{b^u}]$ contains only chronons that are included in the interval $[\mathbf{b^l},\mathbf{b^u}]$. And, finally, sub-interval 3 defined as $[\mathbf{b^l},\mathbf{a^u}]$ contains chronons that are common to both of the intervals. It is clear that any chronon from sub-interval 1 is "before" any chronon from interval $[\mathbf{b^l},\mathbf{b^u}]$. Then the sum of the probabilities of all pairs of the chronons (x,y), where $x\in[\mathbf{a^l},\mathbf{b^l}\text{-}1]$ and $y\in[\mathbf{b^l},\mathbf{b^u}]$, which can be found as

$P_1(<)=\sum_{\mathbf{a}=\mathbf{a}^l}^{\mathbf{b}^l-1}\left\{\mathbf{f}_1(\mathbf{a})\times\sum_{\mathbf{b}=\mathbf{b}^l}^{\mathbf{b}^u}\mathbf{f}_2(\mathbf{b})\right\}$, contributes to the probability $\mathbf{e}^<$. Since $\sum_{\mathbf{b}=\mathbf{b}^l}^{\mathbf{b}^u}\mathbf{f}_2(\mathbf{b})=1$, the probability $P_1(<)$ is equal to $\sum_{\mathbf{a}=\mathbf{a}^l}^{\mathbf{b}^l-1}\mathbf{f}_1(\mathbf{a})$. Similarly, any chronon from the interval $[\mathbf{a}^l,\mathbf{a}^u]$ is "before" any chronon from sub-interval 2. The sum of the probabilities of all pairs of chronons (x,y), where $x\in[\mathbf{a}^l,\mathbf{a}^u]$ and $y\in[\mathbf{a}^u+1,\mathbf{b}^u]$ is $P_2(<)=\sum_{\mathbf{b}=\mathbf{a}^u+1}^{\mathbf{b}^u}\mathbf{f}_2(\mathbf{b})$. The probabilities $P_1(<)$ and $P_2(<)$ take into account several pairs of the values of **a** and **b** twice, which is shown in Table 1 (pairs taken into account twice are underlined).

Table 1: Pairs of values of **a** and **b** taken into account within $P_1(<)$ and $P_2(<)$

Probability	Pairs of values taken into account
$P_1(<)$	(1,3) (1,4) (1,5) (1,6) (1,7) (1,8) (1,9) (2,3) (2,4) (2,5) (2,6) (2,7) (2,8) (2,9)
$P_2(<)$	(1,6) (1,7) (1,8) (1,9) (2,6) (2,7) (2,8) (2,9) (3,6) (3,7) (3,8) (3,9) (4,6) (4,7) (4,8) (4,9) (5,6) (5,7) (5,8) (5,9)

Therefore, the sum of the probabilities of the pairs taken into account twice $P_1(<)\times P_2(<)$ needs be subtracted from $\mathbf{e}^<$. Finally, within sub-interval 3 only pairs of the values of **a** and **b**, where **a**<**b**, contribute to $\mathbf{e}^<$.

We considered above only one kind of situation of overlapping of the intervals of indeterminacy $[\mathbf{a}^l,\mathbf{a}^u]$ and $[\mathbf{b}^l,\mathbf{b}^u]$, particularly when $\mathbf{a}^l<\mathbf{b}^l$, $\mathbf{a}^u>\mathbf{b}^l$, and $\mathbf{a}^u<\mathbf{b}^u$. To cover other overlapping situations as well in the final formula for the probability $\mathbf{e}^<$ (2) we used the notation $[\mathbf{c}^l,\mathbf{c}^u]$ standing for subinterval 3.

$$\mathbf{e}^< = \sum_{\mathbf{a}=\mathbf{a}^l}^{\mathbf{c}^l-1}\mathbf{f}_1(\mathbf{a}) + \sum_{\mathbf{b}=\mathbf{c}^u+1}^{\mathbf{b}^u}\mathbf{f}_2(\mathbf{b}) - \sum_{\mathbf{a}=\mathbf{a}^l}^{\mathbf{c}^l-1}\left\{\mathbf{f}_1(\mathbf{a})\times\sum_{\mathbf{b}=\mathbf{c}^u+1}^{\mathbf{b}^u}\mathbf{f}_2(\mathbf{b})\right\} + \sum_{\mathbf{a}=\mathbf{c}^l}^{\mathbf{c}^u-1}\left\{\mathbf{f}_1(\mathbf{a})\times\sum_{\mathbf{b}=\mathbf{a}+1}^{\mathbf{c}^u}\mathbf{f}_2(\mathbf{b})\right\}, \quad (2)$$

where $\mathbf{c}^l = \max(\mathbf{a}^l, \mathbf{b}^l)$ and $\mathbf{c}^u = \min(\mathbf{a}^u, \mathbf{b}^u)$.

The algorithmic notation of the formula (2) is represented in Figure 4.

$P_1(<) = 0$; $P_2(<) = 0$; $P_3(<) = 0$;
// computing the probability $P_1(<)$
for a=$\mathbf{a}^l$ **to** $\mathbf{c}^l$-1 **do**
  $P_1(<) = P_1(<) + f_1(\mathbf{a})$;
// computing the probability $P_2(<)$
for b=$\mathbf{c}^u$+1 **to** $\mathbf{b}^u$ **do**
  $P_2(<) = P_2(<) + f_2(\mathbf{b})$;
// computing the probability $P_3(<)$
for a=$\mathbf{c}^l$ **to** $\mathbf{c}^u$-1 **do**
  for b=**a**+1 **to** $\mathbf{c}^u$ **do**
    $P_3(<) = P_3(<) + f_1(\mathbf{a})\times f_2(\mathbf{b})$;
// computing the probability $\mathbf{e}^<$
$\mathbf{e}^< = P_1(<) + P_2(<) - P_1(<)\times P_2(<) + P_3(<)$;

Figure 4: Algorithm for computing the probability $\mathbf{e}^<$ between two indeterminate temporal points

In a similar way we can compose the formulas for the probabilities of the temporal relations "after" and "at the same time" between two indeterminate temporal points, represented by formulas (3) and (4) correspondingly:

$$\mathbf{e}^{>} = \sum_{\mathbf{b}=\mathbf{b}^{\mathbf{l}}}^{\mathbf{c}^{\mathbf{l}}-1} \mathbf{f}_2(\mathbf{b}) + \sum_{\mathbf{a}=\mathbf{c}^{\mathbf{u}}+1}^{\mathbf{a}^{\mathbf{u}}} \mathbf{f}_1(\mathbf{a}) - \sum_{\mathbf{b}=\mathbf{b}^{\mathbf{l}}}^{\mathbf{c}^{\mathbf{l}}-1} \left\{ \mathbf{f}_2(\mathbf{b}) \times \sum_{\mathbf{a}=\mathbf{c}^{\mathbf{u}}+1}^{\mathbf{a}^{\mathbf{u}}} \mathbf{f}_1(\mathbf{a}) \right\} + \sum_{\mathbf{b}=\mathbf{c}^{\mathbf{l}}}^{\mathbf{c}^{\mathbf{u}}-1} \left\{ \mathbf{f}_2(\mathbf{b}) \times \sum_{\mathbf{a}=\mathbf{b}+1}^{\mathbf{c}^{\mathbf{u}}} \mathbf{f}_1(\mathbf{a}) \right\} ; \quad (3)$$

$$\mathbf{e}^{=} = \sum_{\mathbf{a}=\mathbf{c}^{\mathbf{l}}}^{\mathbf{c}^{\mathbf{u}}} \left\{ \mathbf{f}_1(\mathbf{a}) \times \mathbf{f}_2(\mathbf{a}) \right\} . \quad (4)$$

A sum of the obtained values of $\mathbf{e}^{<}$, $\mathbf{e}^{=}$, and $\mathbf{e}^{>}$ is equal to 1, since they include all the possible combinations of the values of **a** and **b** when $\sum_{\mathbf{a}=\mathbf{a}^{\mathbf{l}}}^{\mathbf{a}^{\mathbf{u}}} \mathbf{f}_1(\mathbf{a}) = 1$ and $\sum_{\mathbf{b}=\mathbf{b}^{\mathbf{l}}}^{\mathbf{b}^{\mathbf{u}}} \mathbf{f}_2(\mathbf{b}) = 1$.

Further in the paper we compose formulas for the probabilities of Allen's interval relations using the probabilities of the basic relations between the endpoints of the intervals. The probabilities of Allen's relations should take into account possible dependencies among the values of the relations between the endpoints, which we consider in the next section.

4 Relations between the Endpoints of Two Indeterminate Intervals

In this section we analyze the values of the relations between the endpoints of two indeterminate intervals and distinguish the cases when the values of these relations are dependent. These cases are further used to compose the conditional probabilities of the endpoint relations.

Each of the four relations $\mathbf{r}_{11}$, $\mathbf{r}_{12}$, $\mathbf{r}_{21}$, and $\mathbf{r}_{22}$ between the endpoints of two intervals can take three possible values "<", "=", and ">". Let us consider all possible combinations of the values of these relations (Table 2).

Table 2: Combinations of possible values of the relations $\mathbf{r}_{11}$, $\mathbf{r}_{12}$, $\mathbf{r}_{21}$, and $\mathbf{r}_{22}$

	A	B	C	D	E	F	G	H	I
1	< < < < **b**	< = < <	< > < <	= < < <	= = < <	= > < <	> < < <	> = < <	> > < <
2	< < < =	< = < =	< > < =	= < < =	= = < =	= > < =	> < < =	> = < =	> > < =
3	< < < >	< = < >	< > < >	= < < >	= = < >	= > < >	> < < >	> = < >	> > < >
4	< < = < **m**	< = = <	< > = <	= < = <	= = = <	= > = <	> < = <	> = = <	> > = <
5	< < = =	< = = =	< > = =	= < = =	= = = =	= > = =	> < = =	> = = =	> > = =
6	< < = >	< = = >	< > = >	= < = >	= = = >	= > = >	> < = >	> = = >	> > = >
7	< < > < **o**	< = > <	< > > <	= < > < **s**	= = > <	= > > <	> < > < **d**	> = > <	> > > <
8	< < > = **fi**	< = > =	< > > =	= < > = **eq**	= = > =	= > > =	> < > = **f**	> = > =	> > > =
9	< < > > **di**	< = > >	< > > >	= < > > **si**	= = > >	= > > >	> < > > **oi**	> = > > **mi**	> > > > **bi**

The combinations from Table 2 are divided into groups A,B,C,D,E,F,G,H, and I with 9 combinations within each group, according to the values of the relations $\mathbf{r_{11}}$ and $\mathbf{r_{12}}$. Each group is divided into three sections with three combinations in each, depending on the values of $\mathbf{r_{21}}$ and $\mathbf{r_{22}}$. Among those 81 combinations only 13 correspond to the valid Allen's interval relations (shown with a grey background). For example, the combination "<<<<" corresponds to the relation "before". The rest of the combinations are invalid, because in that cases the values of the relations contradict the definition of a temporal interval.

In some cases the values of the relations $\mathbf{r_{11}}$, $\mathbf{r_{12}}$, $\mathbf{r_{21}}$, and $\mathbf{r_{22}}$ are dependent, i.e. if one of them takes a particular value, then another one also takes some particular value. This means that we are dealing with a system of dependent relations between the endpoints of the intervals. Let us consider the cases when the values of the relations between the endpoints are dependent. Let us define a set of the events $\mathbf{\Omega_1} = \{\mathbf{r}_{11}^{<}, \mathbf{r}_{11}^{=}, \mathbf{r}_{11}^{>}, \mathbf{r}_{12}^{<}, \mathbf{r}_{12}^{=}, \mathbf{r}_{12}^{>}, \mathbf{r}_{21}^{<}, \mathbf{r}_{21}^{=}, \mathbf{r}_{21}^{>}, \mathbf{r}_{22}^{<}, \mathbf{r}_{22}^{=}, \mathbf{r}_{22}^{>}\}$, where each event represents the situation when one of the four relations between the endpoints takes a particular temporal value. For each of 12 events from $\mathbf{\Omega_1}$ we define possible values of the relations $\mathbf{r_{11}}$, $\mathbf{r_{12}}$, $\mathbf{r_{21}}$, and $\mathbf{r_{22}}$ (Table 3).

Table 3: Possible values of the relations $\mathbf{r_{11}}$, $\mathbf{r_{12}}$, $\mathbf{r_{21}}$, and $\mathbf{r_{22}}$ for the events from $\mathbf{\Omega_1}$

	$\mathbf{r}_{11}^{<}$	$\mathbf{r}_{11}^{=}$	$\mathbf{r}_{11}^{>}$	$\mathbf{r}_{12}^{<}$	$\mathbf{r}_{12}^{=}$	$\mathbf{r}_{12}^{>}$	$\mathbf{r}_{21}^{<}$	$\mathbf{r}_{21}^{=}$	$\mathbf{r}_{21}^{>}$	$\mathbf{r}_{22}^{<}$	$\mathbf{r}_{22}^{=}$	$\mathbf{r}_{22}^{>}$
$\mathbf{r_{11}}$				?	>	>	<	<	?	?	?	?
$\mathbf{r_{12}}$	<	<	?				<	<	?	<	<	?
$\mathbf{r_{21}}$	?	>	>	?	>	>				?	>	>
$\mathbf{r_{22}}$	?	?	?	?	>	>	<	<	?			

Twelve columns of Table 3 include 12 events from $\mathbf{\Omega_1}$. For example, for the event "$\mathbf{r}_{11}^{<}$" the values of the relations $\mathbf{r_{12}}$, $\mathbf{r_{21}}$, and $\mathbf{r_{22}}$ are "<", "?", and "?" correspondingly. The question mark "?" means that the value of the relation can be any of the three basic relations.

Let us consider the composition of the cases from Table 3.

($\mathbf{r}_{11}^{<}$) The value of the relation $\mathbf{r_{11}}$ is "<", and this means that $\mathbf{s_1<s_2}$. According to Definition 3, $\mathbf{s_1<e_1}$ and $\mathbf{s_2<e_2}$. In this case, the value of the relation $\mathbf{r_{12}}$ is "<" ($\mathbf{s_1<e_2}$). The values of the relations $\mathbf{r_{21}}$ and $\mathbf{r_{22}}$ do not depend on $\mathbf{r_{11}}$ in this case. All combinations from groups B and C (Table 2) are invalid, because they contradict the condition $\mathbf{s_1<e_2}$.

($\mathbf{r}_{11}^{=}$) The value of the relation $\mathbf{r_{11}}$ is "=", and this means that $\mathbf{s_1=s_2}$. Because of $\mathbf{s_1<e_1}$ and $\mathbf{s_2<e_2}$, the values of the relations $\mathbf{r_{12}}$ and $\mathbf{r_{21}}$ are "<" and ">" correspondingly ($\mathbf{s_2<e_1}$ and $\mathbf{s_1<e_2}$). The value of $\mathbf{r_{22}}$ does not depend on $\mathbf{r_{11}}$ in this case. All combinations from groups E and F, and the combinations 1,2,3,4,5, and 6 from group D are invalid.

($\mathbf{r}_{11}^{>}$) The value of the relation $\mathbf{r_{11}}$ is ">", and this means that $\mathbf{s_1>s_2}$. Because of $\mathbf{s_1<e_1}$, the value of the relation $\mathbf{r_{21}}$ is ">" ($\mathbf{e_1>s_2}$). The values of the relations $\mathbf{r_{12}}$ and $\mathbf{r_{22}}$ do not depend on $\mathbf{r_{11}}$ in this case. The combinations 1,2,3,4,5, and 6 from groups G, H and I are invalid.

($\mathbf{r}_{12}^{<}$) The value of the relation $\mathbf{r_{12}}$ is "<", and this means that $\mathbf{s_1<e_2}$. The values of the relations $\mathbf{r_{11}}$ $\mathbf{r_{21}}$ and $\mathbf{r_{22}}$ do not depend on $\mathbf{r_{12}}$ in this case.

($\mathbf{r}_{12}^{=}, \mathbf{r}_{12}^{>}$) The value of the relation $\mathbf{r_{12}}$ is "=" or ">", and this means that $\mathbf{s_1 \geq e_2}$. Because of $\mathbf{s_1<e_1}$ and $\mathbf{s_2<e_2}$, the values of the relations $\mathbf{r_{11}}$, $\mathbf{r_{21}}$ and $\mathbf{r_{21}}$ are ">" ($\mathbf{s_1>s_2}$, $\mathbf{e_1>s_1}$ and $\mathbf{e_1>e_2}$). Groups B, C, E, F, H, and I, except the combination 9 from groups H and I, are invalid.

($r^<_{21}, r^=_{21}$) The value of the relation $\mathbf{r}_{21}$ is "<" or "=", and this means that $\mathbf{e}_1 \leq \mathbf{s}_2$. Because of $\mathbf{s}_1 < \mathbf{e}_1$ and $\mathbf{s}_2 < \mathbf{e}_2$, the values of the relations $\mathbf{r}_{11}$, $\mathbf{r}_{12}$ and $\mathbf{r}_{22}$ are "<" ($\mathbf{s}_1 < \mathbf{s}_2$, $\mathbf{s}_1 < \mathbf{e}_2$ and $\mathbf{e}_1 < \mathbf{e}_2$). The combinations 2,3,5, and 6 from group A and the combinations 1,2,3,4,5, and 6 from groups B, C, D, E, F, G, H, and I are invalid.

($r^>_{21}$) The value of the relation $\mathbf{r}_{21}$ is ">", and this means that $\mathbf{e}_1 > \mathbf{s}_2$. The values of the relations $\mathbf{r}_{11}$ $\mathbf{r}_{12}$ and $\mathbf{r}_{22}$ do not depend on $\mathbf{r}_{21}$ in this case.

($r^<_{22}$) The value of the relation $\mathbf{r}_{22}$ is "<", and this means that $\mathbf{e}_1 < \mathbf{e}_2$. Because of $\mathbf{s}_1 < \mathbf{e}_1$, the value of the relation $\mathbf{r}_{12}$ is "<" ($\mathbf{s}_1 < \mathbf{e}_2$). The values of the relations $\mathbf{r}_{11}$ and $\mathbf{r}_{21}$ do not depend on $\mathbf{r}_{22}$ in this case. The combinations 1, 4, and 7 from groups B, C, E, F, H, and I are invalid.

($r^=_{22}$) The value of the relation $\mathbf{r}_{22}$ is "=", and this means that $\mathbf{e}_1 = \mathbf{e}_2$. Because of $\mathbf{s}_1 < \mathbf{e}_1$ and $\mathbf{s}_2 < \mathbf{e}_2$, the values of the relations $\mathbf{r}_{12}$ and $\mathbf{r}_{21}$ are "<" and ">" correspondingly ($\mathbf{s}_1 < \mathbf{e}_2$ and $\mathbf{e}_1 > \mathbf{s}_2$). The value of the relation $\mathbf{r}_{11}$ does not depend on $\mathbf{r}_{22}$ in this case. The combinations 2 and 5 from groups A, D, and G, and the combinations 2,5, and 8 from groups B, C, E, F, H, and I are invalid.

($r^>_{22}$) The value of the relation $\mathbf{r}_{22}$ is ">", and this means that $\mathbf{e}_1 > \mathbf{e}_2$. Because of $\mathbf{s}_2 < \mathbf{e}_2$, the value of the relation $\mathbf{r}_{21}$ is ">" ($\mathbf{e}_1 > \mathbf{s}_2$). The values of the relations $\mathbf{r}_{11}$ and $\mathbf{r}_{12}$ do not depend on $\mathbf{r}_{22}$ in this case. The combinations 3 and 6 from groups A, D, and G, and the combinations 3,6, and 9 from groups B, C, E, F, H, and I are invalid.

The considered above cases together make invalid all combinations from Table 2, except those that are shown with a grey background and represent the valid Allen's interval relations.

In the next section we compose the conditional probabilities of the relations between the endpoints using Table 3, and then compose the probabilities of Allen's relations using the obtained conditional probabilities.

5 Probabilities of Allen's Interval Relations

In this section we compose formulas for the probabilities of Allen's interval relations. These probabilities are calculated using the conditional probabilities of the relations between the endpoints of the intervals.

The main concept used throughout this section is the notion of conditional probability. A conditional probability P(A|B) denotes the probability of an event A, calculated under assumption that an event B occurred. If A does not depend on B, the conditional probability P(A|B) is transformed into an ordinary probability P(A). In the case of four events A, B, C, and D, the conditional probability P(A|BCD) is the probability of A calculated under assumption that the events B, C, and D occurred.

Since we are dealing with a system of dependent relations, as it was shown in the Section 4, the probability of the relation between two endpoints should take into account the values of other three relations between the endpoints. It means, that the probability of each of the relations $\mathbf{r}_{11}$, $\mathbf{r}_{12}$, $\mathbf{r}_{21}$, and $\mathbf{r}_{22}$ is the conditional probability of this relation under some particular values of the other three relations.

We are only interested in those combinations of possible values of the relations $\mathbf{r}_{11}$, $\mathbf{r}_{12}$, $\mathbf{r}_{21}$, and $\mathbf{r}_{22}$ that correspond to the thirteen Allen's interval relations. These combinations were presented in Table 2 in Section 4. For each of the four relations between the endpoints we compose below a table including the values of the conditional probabilities of this relation. So, Table 4 presents the values of the conditional probabilities of the relation $\mathbf{r}_{11}$.

Table 4: Conditional probabilities of $\mathbf{r}_{11}$

	$\mathbf{r}_{12}^{<}\mathbf{r}_{21}^{<}\mathbf{r}_{22}^{<}$	$\mathbf{r}_{12}^{<}\mathbf{r}_{21}^{=}\mathbf{r}_{22}^{<}$	$\mathbf{r}_{12}^{<}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{<}$	$\mathbf{r}_{12}^{<}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{=}$	$\mathbf{r}_{12}^{<}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{>}$	$\mathbf{r}_{12}^{=}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{>}$	$\mathbf{r}_{12}^{>}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{>}$
$P(\mathbf{r}_{11}^{<} \mid \mathbf{r}_{12}\mathbf{r}_{21}\mathbf{r}_{22})$	1	1	$\mathbf{e}_{11}^{<}$	$\mathbf{e}_{11}^{<}$	$\mathbf{e}_{11}^{<}$	0	0
$P(\mathbf{r}_{11}^{=} \mid \mathbf{r}_{12}\mathbf{r}_{21}\mathbf{r}_{22})$	0	0	$\mathbf{e}_{11}^{=}$	$\mathbf{e}_{11}^{=}$	$\mathbf{e}_{11}^{=}$	0	0
$P(\mathbf{r}_{11}^{>} \mid \mathbf{r}_{12}\mathbf{r}_{21}\mathbf{r}_{22})$	0	0	$\mathbf{e}_{11}^{>}$	$\mathbf{e}_{11}^{>}$	$\mathbf{e}_{11}^{>}$	1	1

Let us consider how the probability values in Table 4 were obtained using as an example the values of the probability $P(\mathbf{r}_{11}^{<} \mid \mathbf{r}_{12}\mathbf{r}_{21}\mathbf{r}_{22})$. When the value of the relation $\mathbf{r}_{11}$ depends on at least one of the relations $\mathbf{r}_{12}$, $\mathbf{r}_{21}$, or $\mathbf{r}_{22}$, and this value is "<" (Table 3, Section 4), the conditional probability of $\mathbf{r}_{11}$ is equal to 1. When the value of $\mathbf{r}_{11}$ does not depend on any value of the relations $\mathbf{r}_{12}$, $\mathbf{r}_{21}$, or $\mathbf{r}_{22}$, the probability $P(\mathbf{r}_{11}^{<} \mid \mathbf{r}_{12}\mathbf{r}_{21}\mathbf{r}_{22})$ is transformed into $P(\mathbf{r}_{11}^{<})$, which is defined by the probability value $\mathbf{e}_{11}^{<}$ from the vector $\left(\mathbf{e}^{<},\mathbf{e}^{=},\mathbf{e}^{>}\right)_{11}$ in the matrix $\mathfrak{R}$. And, finally, we suppose that the probability $P(\mathbf{r}_{11}^{<} \mid \mathbf{r}_{12}\mathbf{r}_{21}\mathbf{r}_{22})$ is equal to 0 when the combination of the events does not correspond to any of Allen's relations. In a similar way we can compose tables of the values of the conditional probabilities of the relations $\mathbf{r}_{12}$ (Table 5), $\mathbf{r}_{21}$ (Table 6), and $\mathbf{r}_{22}$ (Table 7).

Table 5: Conditional probabilities of $\mathbf{r}_{12}$

	$\mathbf{r}_{11}^{<}\mathbf{r}_{21}^{<}\mathbf{r}_{22}^{<}$, $\mathbf{r}_{11}^{<}\mathbf{r}_{21}^{=}\mathbf{r}_{22}^{<}$, $\mathbf{r}_{11}^{<}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{<}$, $\mathbf{r}_{11}^{<}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{=}$, $\mathbf{r}_{11}^{<}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{>}$, $\mathbf{r}_{11}^{=}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{<}$, $\mathbf{r}_{11}^{=}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{=}$, $\mathbf{r}_{11}^{=}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{>}$, $\mathbf{r}_{11}^{>}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{<}$, $\mathbf{r}_{11}^{>}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{=}$	$\mathbf{r}_{11}^{>}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{>}$
$P(\mathbf{r}_{12}^{<} \mid \mathbf{r}_{11}\mathbf{r}_{21}\mathbf{r}_{22})$	1	$\mathbf{e}_{12}^{<}$
$P(\mathbf{r}_{12}^{=} \mid \mathbf{r}_{11}\mathbf{r}_{21}\mathbf{r}_{22})$	0	$\mathbf{e}_{12}^{=}$
$P(\mathbf{r}_{12}^{>} \mid \mathbf{r}_{11}\mathbf{r}_{21}\mathbf{r}_{22})$	0	$\mathbf{e}_{12}^{>}$

Table 6: Conditional probabilities of $\mathbf{r}_{21}$

	$\mathbf{r}_{11}^{<}\mathbf{r}_{12}^{<}\mathbf{r}_{22}^{<}$	$\mathbf{r}_{11}^{<}\mathbf{r}_{12}^{<}\mathbf{r}_{22}^{=}$, $\mathbf{r}_{11}^{<}\mathbf{r}_{12}^{<}\mathbf{r}_{22}^{>}$, $\mathbf{r}_{11}^{=}\mathbf{r}_{12}^{<}\mathbf{r}_{22}^{<}$, $\mathbf{r}_{11}^{=}\mathbf{r}_{12}^{<}\mathbf{r}_{22}^{=}$, $\mathbf{r}_{11}^{=}\mathbf{r}_{12}^{<}\mathbf{r}_{22}^{>}$, $\mathbf{r}_{11}^{>}\mathbf{r}_{12}^{<}\mathbf{r}_{22}^{<}$, $\mathbf{r}_{11}^{>}\mathbf{r}_{12}^{<}\mathbf{r}_{22}^{=}$, $\mathbf{r}_{11}^{>}\mathbf{r}_{12}^{<}\mathbf{r}_{22}^{>}$, $\mathbf{r}_{11}^{>}\mathbf{r}_{12}^{=}\mathbf{r}_{22}^{>}$, $\mathbf{r}_{11}^{>}\mathbf{r}_{12}^{>}\mathbf{r}_{22}^{>}$
$P(\mathbf{r}_{21}^{<} \mid \mathbf{r}_{11}\mathbf{r}_{12}\mathbf{r}_{22})$	$\mathbf{e}_{21}^{<}$	0
$P(\mathbf{r}_{21}^{=} \mid \mathbf{r}_{11}\mathbf{r}_{12}\mathbf{r}_{22})$	$\mathbf{e}_{21}^{=}$	1
$P(\mathbf{r}_{21}^{>} \mid \mathbf{r}_{11}\mathbf{r}_{12}\mathbf{r}_{22})$	$\mathbf{e}_{21}^{>}$	1

Table 7: Conditional probabilities of $\mathbf{r}_{22}$

	$\mathbf{r}_{21}^{<}\mathbf{r}_{12}^{<}\mathbf{r}_{11}^{<}$	$\mathbf{r}_{21}^{=}\mathbf{r}_{12}^{<}\mathbf{r}_{11}^{<}$	$\mathbf{r}_{21}^{>}\mathbf{r}_{12}^{<}\mathbf{r}_{11}^{<}$	$\mathbf{r}_{21}^{>}\mathbf{r}_{12}^{<}\mathbf{r}_{11}^{=}$	$\mathbf{r}_{21}^{>}\mathbf{r}_{12}^{<}\mathbf{r}_{11}^{>}$	$\mathbf{r}_{21}^{>}\mathbf{r}_{12}^{=}\mathbf{r}_{11}^{>}$	$\mathbf{r}_{21}^{>}\mathbf{r}_{12}^{>}\mathbf{r}_{11}^{>}$
$P(\mathbf{r}_{22}^{<} \mid \mathbf{r}_{21}\mathbf{r}_{12}\mathbf{r}_{11})$	1	1	$\mathbf{e}_{22}^{<}$	$\mathbf{e}_{22}^{<}$	$\mathbf{e}_{22}^{<}$	0	0
$P(\mathbf{r}_{22}^{=} \mid \mathbf{r}_{21}\mathbf{r}_{12}\mathbf{r}_{11})$	0	0	$\mathbf{e}_{22}^{=}$	$\mathbf{e}_{22}^{=}$	$\mathbf{e}_{22}^{=}$	0	0
$P(\mathbf{r}_{22}^{>} \mid \mathbf{r}_{21}\mathbf{r}_{12}\mathbf{r}_{11})$	0	0	$\mathbf{e}_{22}^{>}$	$\mathbf{e}_{22}^{>}$	$\mathbf{e}_{22}^{>}$	1	1

The probability P(eq) of Allen's relation "equals", defined by formula (5) is a multiplication of the four conditional probabilities of the relations $\mathbf{r_{11}}$, $\mathbf{r_{12}}$, $\mathbf{r_{21}}$, and $\mathbf{r_{22}}$ when the events $\mathbf{r}_{11}^{=}$, $\mathbf{r}_{12}^{<}$, $\mathbf{r}_{21}^{>}$, and $\mathbf{r}_{22}^{=}$ from $\mathbf{\Omega_1}$ occur simultaneously:

$$\mathrm{P(eq)} = \mathbf{P}\left(\mathbf{r}_{11}^{=} \middle| \mathbf{r}_{12}^{<}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{=}\right) \mathbf{P}\left(\mathbf{r}_{12}^{<} \middle| \mathbf{r}_{11}^{=}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{=}\right) \mathbf{P}\left(\mathbf{r}_{21}^{>} \middle| \mathbf{r}_{11}^{=}\mathbf{r}_{12}^{<}\mathbf{r}_{22}^{=}\right) \mathbf{P}\left(\mathbf{r}_{22}^{=} \middle| \mathbf{r}_{11}^{=}\mathbf{r}_{12}^{<}\mathbf{r}_{21}^{>}\right) = \mathbf{e}_{11}^{=}\ \mathbf{e}_{22}^{=}\,. \tag{5}$$

In formula (5) the values of the conditional probabilities are taken from Tables 4,5,6, and 7. In a similar way we can compose the probabilities of other Allen's relations:

$$\mathrm{P(b)} = \mathbf{P}\left(\mathbf{r}_{11}^{<} \middle| \mathbf{r}_{12}^{<}\mathbf{r}_{21}^{<}\mathbf{r}_{22}^{<}\right) \mathbf{P}\left(\mathbf{r}_{12}^{<} \middle| \mathbf{r}_{11}^{<}\mathbf{r}_{21}^{<}\mathbf{r}_{22}^{<}\right) \mathbf{P}\left(\mathbf{r}_{21}^{<} \middle| \mathbf{r}_{11}^{<}\mathbf{r}_{12}^{<}\mathbf{r}_{22}^{<}\right) \mathbf{P}\left(\mathbf{r}_{22}^{<} \middle| \mathbf{r}_{11}^{<}\mathbf{r}_{12}^{<}\mathbf{r}_{21}^{<}\right) = \mathbf{e}_{11}^{<}\ \mathbf{e}_{21}^{<}\,; \tag{6}$$

$$\mathrm{P(bi)} = \mathbf{P}\left(\mathbf{r}_{11}^{>} \middle| \mathbf{r}_{12}^{>}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{>}\right) \mathbf{P}\left(\mathbf{r}_{12}^{>} \middle| \mathbf{r}_{11}^{>}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{>}\right) \mathbf{P}\left(\mathbf{r}_{21}^{>} \middle| \mathbf{r}_{11}^{>}\mathbf{r}_{12}^{>}\mathbf{r}_{22}^{>}\right) \mathbf{P}\left(\mathbf{r}_{22}^{>} \middle| \mathbf{r}_{11}^{>}\mathbf{r}_{12}^{>}\mathbf{r}_{21}^{>}\right) = \mathbf{e}_{11}^{>}\ \mathbf{e}_{12}^{>}\,; \tag{7}$$

$$\mathrm{P(d)} = \mathbf{P}\left(\mathbf{r}_{11}^{>} \middle| \mathbf{r}_{12}^{<}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{<}\right) \mathbf{P}\left(\mathbf{r}_{12}^{<} \middle| \mathbf{r}_{11}^{>}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{<}\right) \mathbf{P}\left(\mathbf{r}_{21}^{>} \middle| \mathbf{r}_{11}^{>}\mathbf{r}_{12}^{<}\mathbf{r}_{22}^{<}\right) \mathbf{P}\left(\mathbf{r}_{22}^{<} \middle| \mathbf{r}_{11}^{>}\mathbf{r}_{12}^{<}\mathbf{r}_{21}^{>}\right) = \mathbf{e}_{11}^{>}\ \mathbf{e}_{12}^{<}\ \mathbf{e}_{22}^{<}\,; \tag{8}$$

$$\mathrm{P(di)} = \mathbf{P}\left(\mathbf{r}_{11}^{<} \middle| \mathbf{r}_{12}^{<}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{>}\right) \mathbf{P}\left(\mathbf{r}_{12}^{<} \middle| \mathbf{r}_{11}^{<}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{>}\right) \mathbf{P}\left(\mathbf{r}_{21}^{>} \middle| \mathbf{r}_{11}^{<}\mathbf{r}_{12}^{<}\mathbf{r}_{22}^{>}\right) \mathbf{P}\left(\mathbf{r}_{22}^{>} \middle| \mathbf{r}_{11}^{<}\mathbf{r}_{12}^{<}\mathbf{r}_{21}^{>}\right) = \mathbf{e}_{11}^{<}\ \mathbf{e}_{21}^{>}\ \mathbf{e}_{22}^{>}\,; \tag{9}$$

$$\mathrm{P(o)} = \mathbf{P}\left(\mathbf{r}_{11}^{<} \middle| \mathbf{r}_{12}^{<}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{<}\right) \mathbf{P}\left(\mathbf{r}_{12}^{<} \middle| \mathbf{r}_{11}^{<}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{<}\right) \mathbf{P}\left(\mathbf{r}_{21}^{>} \middle| \mathbf{r}_{11}^{<}\mathbf{r}_{12}^{<}\mathbf{r}_{22}^{<}\right) \mathbf{P}\left(\mathbf{r}_{22}^{<} \middle| \mathbf{r}_{11}^{<}\mathbf{r}_{12}^{<}\mathbf{r}_{21}^{>}\right) = \mathbf{e}_{11}^{<}\ \mathbf{e}_{21}^{>}\ \mathbf{e}_{22}^{<}\,; \tag{10}$$

$$\mathrm{P(oi)} = \mathbf{P}\left(\mathbf{r}_{11}^{>} \middle| \mathbf{r}_{12}^{<}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{>}\right) \mathbf{P}\left(\mathbf{r}_{12}^{<} \middle| \mathbf{r}_{11}^{>}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{>}\right) \mathbf{P}\left(\mathbf{r}_{21}^{>} \middle| \mathbf{r}_{11}^{>}\mathbf{r}_{12}^{<}\mathbf{r}_{22}^{>}\right) \mathbf{P}\left(\mathbf{r}_{22}^{>} \middle| \mathbf{r}_{11}^{>}\mathbf{r}_{12}^{<}\mathbf{r}_{21}^{>}\right) = \mathbf{e}_{11}^{>}\ \mathbf{e}_{12}^{<}\ \mathbf{e}_{22}^{>}\,; \tag{11}$$

$$\mathrm{P(m)} = \mathbf{P}\left(\mathbf{r}_{11}^{<} \middle| \mathbf{r}_{12}^{<}\mathbf{r}_{21}^{=}\mathbf{r}_{22}^{<}\right) \mathbf{P}\left(\mathbf{r}_{12}^{<} \middle| \mathbf{r}_{11}^{<}\mathbf{r}_{21}^{=}\mathbf{r}_{22}^{<}\right) \mathbf{P}\left(\mathbf{r}_{21}^{=} \middle| \mathbf{r}_{11}^{<}\mathbf{r}_{12}^{<}\mathbf{r}_{22}^{<}\right) \mathbf{P}\left(\mathbf{r}_{22}^{<} \middle| \mathbf{r}_{11}^{<}\mathbf{r}_{12}^{<}\mathbf{r}_{21}^{=}\right) = \mathbf{e}_{11}^{<}\ \mathbf{e}_{21}^{=}\,; \tag{12}$$

$$\mathrm{P(mi)} = \mathbf{P}\left(\mathbf{r}_{11}^{>} \middle| \mathbf{r}_{12}^{=}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{>}\right) \mathbf{P}\left(\mathbf{r}_{12}^{=} \middle| \mathbf{r}_{11}^{>}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{>}\right) \mathbf{P}\left(\mathbf{r}_{21}^{>} \middle| \mathbf{r}_{11}^{>}\mathbf{r}_{12}^{=}\mathbf{r}_{22}^{>}\right) \mathbf{P}\left(\mathbf{r}_{22}^{>} \middle| \mathbf{r}_{11}^{>}\mathbf{r}_{12}^{=}\mathbf{r}_{21}^{>}\right) = \mathbf{e}_{11}^{>}\ \mathbf{e}_{12}^{=}\,; \tag{13}$$

$$\mathrm{P(s)} = \mathbf{P}\left(\mathbf{r}_{11}^{=} \middle| \mathbf{r}_{12}^{<}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{<}\right) \mathbf{P}\left(\mathbf{r}_{12}^{<} \middle| \mathbf{r}_{11}^{=}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{<}\right) \mathbf{P}\left(\mathbf{r}_{21}^{>} \middle| \mathbf{r}_{11}^{=}\mathbf{r}_{12}^{<}\mathbf{r}_{22}^{<}\right) \mathbf{P}\left(\mathbf{r}_{22}^{<} \middle| \mathbf{r}_{11}^{=}\mathbf{r}_{21}^{>}\mathbf{r}_{12}^{<}\right) = \mathbf{e}_{11}^{=}\ \mathbf{e}_{22}^{<}\,; \tag{14}$$

$$\mathrm{P(si)} = \mathbf{P}\left(\mathbf{r}_{11}^{=} \middle| \mathbf{r}_{12}^{<}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{>}\right) \mathbf{P}\left(\mathbf{r}_{12}^{<} \middle| \mathbf{r}_{11}^{=}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{>}\right) \mathbf{P}\left(\mathbf{r}_{21}^{>} \middle| \mathbf{r}_{11}^{=}\mathbf{r}_{12}^{<}\mathbf{r}_{22}^{>}\right) \mathbf{P}\left(\mathbf{r}_{22}^{>} \middle| \mathbf{r}_{11}^{=}\mathbf{r}_{12}^{<}\mathbf{r}_{21}^{>}\right) = \mathbf{e}_{11}^{=}\ \mathbf{e}_{22}^{>}\,; \tag{15}$$

$$\mathrm{P(f)} = \mathbf{P}\left(\mathbf{r}_{11}^{>} \middle| \mathbf{r}_{12}^{<}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{=}\right) \mathbf{P}\left(\mathbf{r}_{12}^{<} \middle| \mathbf{r}_{11}^{>}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{=}\right) \mathbf{P}\left(\mathbf{r}_{21}^{>} \middle| \mathbf{r}_{11}^{>}\mathbf{r}_{12}^{<}\mathbf{r}_{22}^{=}\right) \mathbf{P}\left(\mathbf{r}_{22}^{=} \middle| \mathbf{r}_{11}^{>}\mathbf{r}_{12}^{<}\mathbf{r}_{21}^{>}\right) = \mathbf{e}_{11}^{>}\ \mathbf{e}_{12}^{<}\ \mathbf{e}_{22}^{=}\,; \tag{16}$$

$$\mathrm{P(fi)} = \mathbf{P}\left(\mathbf{r}_{11}^{<} \middle| \mathbf{r}_{12}^{<}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{>}\right) \mathbf{P}\left(\mathbf{r}_{12}^{<} \middle| \mathbf{r}_{11}^{<}\mathbf{r}_{21}^{>}\mathbf{r}_{22}^{>}\right) \mathbf{P}\left(\mathbf{r}_{21}^{>} \middle| \mathbf{r}_{11}^{<}\mathbf{r}_{12}^{<}\mathbf{r}_{22}^{>}\right) \mathbf{P}\left(\mathbf{r}_{22}^{=} \middle| \mathbf{r}_{11}^{<}\mathbf{r}_{12}^{<}\mathbf{r}_{21}^{>}\right) = \mathbf{e}_{11}^{<}\ \mathbf{e}_{21}^{>}\ \mathbf{e}_{22}^{=}\,. \tag{17}$$

Allen's interval relations are the only thirteen possible relations that can hold between two valid temporal intervals, and at least one of these relations is possible for two valid indeterminate temporal intervals. Therefore, the sum of the probabilities of Allen's relations defined by the above formulas should be equal to 1. The proof of this fact is given in Appendix A.

In the next section we consider an example of using the proposed in the paper estimation mechanism.

6 Example

Let us consider an example of using the proposed estimation mechanism. A tube manufacturing plant makes tubes using steel blanks that are delivered to the plant by warehouses. Let us suppose that at the plant there are two temporal databases with the temporal granularity of 1 day keeping records about manufacturing of tubes and delivery of steel blanks. Each record in the first database includes at least: ”Series” – the identity number for each manufactured series of tubes, “Start of production” - the timestamp defining the start of the production, “End of production” – the timestamp defining the end of the production, and “Defective tubes” - the percentage of defective tubes for a particular manufactured series. A fragment of the first database is represented by Table 8.

Table 8: A fragment of the database keeping records about manufacturing of tubes

Series	Start of production	End of production	Defective tubes (%)
#T10	1 Jan – 3 Jan	15 Jan – 21 Jan	11
#T11	7 Jan – 11 Jan	18 Jan – 22 Jan	5
#T12	3 Feb – 7 Feb	12 Feb – 14 Feb	3

Often the start and the end of the production are indeterminate temporal points. For example, the production of the series #T10 started during the first 3 days of January, and ended during the period from 15-th till 21-st of January. The manufacturing process itself is an indeterminate temporal interval, as it is defined in Section 2.

Each record in the second database includes at least: “Series” - the identity number for each delivered series of blanks, “Supplier” - the warehouse that supplied the series of blanks, “Delivery date” - the timestamp defining when the series of blanks was sent from the warehouse. A fragment of this database is represented by Table 9.

Table 9. A fragment of the database keeping records about delivery of steel blanks

Series #	Sender	Delivery date
#B10	Western	3-4 Jan
#B20	Eastern	9-10 Jan
#B25	Western	25-26 Jan
#B26	Western	2-3 Feb

In Table 9 the delivery dates are indeterminate temporal points. A delivery period is 5-11 days meaning that a particular series of blanks arrived to the plant after 5-11 days after the delivery date specified in the database. For example, the series #B10, that was sent on 3-d or 4-th of January, should arrive to the plant during the period from 9-th till 16-th of January. Therefore, the temporal intervals whose starting point is the delivery date and the end point is a date when blanks arrived to the plant are indeterminate temporal intervals.

When a produced series of tubes has a high percentage of defective tubes (greater than 10) we are interested in which series of blanks was used to manufacture this series of tubes. To answer the question we need to know the temporal relation between the production of tubes temporal interval and the delivery of blanks temporal interval. In many situations we cannot derive a certain relation between these intervals since they are indeterminate, but using the proposed in this paper approach we can estimate the uncertain temporal relation between them. Let us assume that a particular series of steel blanks could be used in the production of a particular series of tubes if blanks arrived to the plant before the production of tubes is finished. This assumption is supported by Allen’s temporal relations “includes”’, “started-by”, “overlapped-by”, “met-by”, and “after”, and the sum of their probabilities P(di), P(si), P(oi), P(mi), and P(bi) is the probability that blanks could be used in the production.

The series of tubes #T10 from Table 8 has the percentage of defective tubes equal to 11. Let us estimate the relation between the production of this series of tubes defined by the interval **A**[1–3, 15–21] and the delivery of the blanks #B10 defined by the interval **B**[3-4, 9–16]. The relation between intervals **A** and **B** is illustrated by Figure 2 (Section 2). Let us suppose, that chronons within the intervals $\mathbf{s_1}$[1-3], $\mathbf{e_1}$[15-21], $\mathbf{s_2}$[3-4], $\mathbf{e_2}$[9-16] are equally probable meaning that the values of the p.m.f.s for the endpoints $\mathbf{s_1}$, $\mathbf{e_1}$, $\mathbf{s_2}$, and $\mathbf{e_2}$ are $\mathbf{f_1}(\mathbf{s_1}) = \frac{1}{3}$, $\mathbf{f_2}(\mathbf{e_1}) = \frac{1}{7}$, $\mathbf{f_3}(\mathbf{s_2}) = 0.5$, and $\mathbf{f_4}(\mathbf{e_2}) = \frac{1}{8}$. Applying formulas (2)-(4) presented in Section 3, and taking into account the p.m.f.s we can calculate the probability values for the vectors in the matrix $\mathfrak{R}$ representing the relation between **A** and **B**, which is $\mathfrak{R} = \begin{bmatrix} \left(\frac{5}{6},\frac{1}{6},0\right)_{11} & \left(1,0,0\right)_{12} \\ \left(0,0,1\right)_{21} & \left(\frac{1}{56},\frac{2}{56},\frac{53}{56}\right)_{22} \end{bmatrix}_{\mathbf{A},\mathbf{B}}$. After that, applying formulas (5)-(17) presented in Section 5 to the probability values from the matrix $\mathfrak{R}$ we can calculate the probabilities of Allen's relations: P(di) = 0.7887, P(si) = 0.1577, P(oi) = 0, P(mi) = 0, and P(bi) = 0. The sum of these probabilities $P_{\mathbf{A,B}}$ = 0.9464, which is the probability that the series of blanks #B10 could be used during the production of the series of tubes #T10.

Let us now estimate the probability that the series of blanks #B20 from Table 9 could be used in the production of the series of tubes #T10. The interval **A**[1–3, 15–21] representing the production of tubes is the same as in the first estimation including the values of the p.m.f.s $\mathbf{f_1}(\mathbf{s_1})$ and $\mathbf{f_2}(\mathbf{e_1})$. The interval **C**[9–10, 15–22] defines the delivery of blanks #B20 to the plant. Here we again suppose that the chronons within the intervals of indeterminacy for the endpoints of **C** are equally probable, i.e. $\mathbf{f_5}(\mathbf{s_2}) = 0.5$, and $\mathbf{f_6}(\mathbf{e_2}) = \frac{1}{8}$. The matrix $\mathfrak{R}$ representing the temporal relation between **A** and **C** is $\mathfrak{R} = \begin{bmatrix} \left(1,0,0\right)_{11} & \left(1,0,0\right)_{12} \\ \left(0,0,1\right)_{21} & \left(\frac{3}{8},\frac{1}{8},\frac{4}{8}\right)_{22} \end{bmatrix}_{\mathbf{A},\mathbf{C}}$.
The probabilities of Allen's relations are: P(di) = 0.5, P(si) = 0, P(oi) = 0, P(mi) = 0, and P(bi) = 0. The sum of these probabilities $P_{\mathbf{A,C}}$ = 0.5, which is the probability that blanks #B20 could be used in the production of tubes #T10.

In this example the derived probability value $P_{\mathbf{A,B}}$ or $P_{\mathbf{A,C}}$ gives us the probability of the situation when the indeterminate intervals **A** and **B** or **A** and **C** are located in such a way when the assumed condition (blanks should arrive before the manufacturing is finished) is fulfilled. The obtained probability value $P_{\mathbf{A,B}}$ = 0.9464 is bigger than the probability value $P_{\mathbf{A,C}}$ = 0.5 meaning that under the known conditions the relation between the intervals **A** and **B** supports better our assumption than the relation between the intervals **A** and **C**. Although, this does not mean at all, that the relation between **A** and **C** does not support the assumed condition, and to be able to select between the present alternatives we need additional information. The main goal of the example was only to show that there are situations where there is a need for estimating uncertain temporal relations, and the proposed in this paper formalism could be applied there.

7 Limitations

In this section we address the major limitations of the proposed in this paper formalism.

The probabilistic approach that was used in the paper is actually one of the means for handling uncertainty, as well as possibility theory, Dempster-Shafer theory, and numerous logical approaches. The method for handling uncertainty was selected reflecting our goals of having numerical measures of uncertainty, and a solid mathematical background behind the method. The probabilistic approach is closely related to statistics which potentially can be used as one of the means of defining the p.m.f. by analyzing the past data.

Our approach assumes full knowledge about the values of the p.m.f., which does not happen very often in real applications. In the case when the distribution is unknown and we have no additional information about it, the uniform distribution is a useful assumption, because we have no reason to favor one chronon over another.

We selected the discrete time model for our formalism, because:

1) it is much more commonly used in the temporal databases area, one of the important application areas for temporal representation and reasoning;
2) the formalisms based on continuous model are more complicated compared to the formalisms based on discrete model.

Generally, the proposed representation can be applied to continuous time model, and we consider it as one possible direction for further research. Certainly there are some domains where the continuous time model is more natural, but more applications are those, where the discrete representation is used.

8 Conclusion

In this paper we proposed an approach to estimate uncertain temporal relations between indeterminate temporal intervals by calculating the probabilities of Allen's interval relations. The uncertain relation between two intervals is represented as a matrix, four elements of which are the relations between the endpoints of these intervals. The relation between two indeterminate points is represented as a vector with three probability values denoting the probabilities of the three basic relations between these points. The probabilities of Allen's relations were composed as joint conditional probabilities of the correspondent relations between the endpoints of the intervals. We considered an example of using the proposed estimation approach, and addressed the major limitations of the work.

Also the proposed estimation mechanism can be used in a query language that supports temporal indeterminacy using probabilities, for example, TSQL2 [23], and hence we did not conceive as a goal the development of a new query language.

As one direction for further research we consider specifying the values of the p.m.f. using available indirect information about it. Moreover, the study of the behavior of the approach in real applications needs additional research.

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Appendix A

The sum of the probabilities of Allen's relations defined by formulas (5)-(17) in Section 5 is equal to 1.

Proof. According to Definition 2 (Section 2), the sum of the probability values $\mathbf{e}^{<}$, $\mathbf{e}^{=}$, and $\mathbf{e}^{>}$ from a vector representing the relation between two indeterminate temporal points is equal to 1. The matrix $\mathfrak{R}$ representing the relation between two indeterminate temporal intervals includes four such vectors. Therefore, we can obtain four equations for the probabilities of the basic relations between the endpoints of the temporal intervals:

$$\mathbf{e}_{11}^{<} + \mathbf{e}_{11}^{=} + \mathbf{e}_{11}^{>} = 1, \tag{18}$$

$$\mathbf{e}_{12}^{<} + \mathbf{e}_{12}^{=} + \mathbf{e}_{12}^{>} = 1, \tag{19}$$

$$\mathbf{e}_{21}^{<} + \mathbf{e}_{21}^{=} + \mathbf{e}_{21}^{>} = 1, \tag{20}$$

$$\mathbf{e}_{22}^{<} + \mathbf{e}_{22}^{=} + \mathbf{e}_{22}^{>} = 1. \tag{21}$$

The sum of the probabilities of Allen's relations defined by formulas (5)-(17) can be transformed taking into account equations (18)-(21):

$$\begin{gathered}
P(eq)+P(b)+P(bi)+P(d)+P(di)+P(o)+P(oi)+P(m)+P(mi)+P(s)+P(si)+P(f)+P(fi)= \\
= \mathbf{e}_{11}^{=}\,\mathbf{e}_{22}^{=} + \mathbf{e}_{11}^{<}\,\mathbf{e}_{21}^{<} + \mathbf{e}_{11}^{>}\,\mathbf{e}_{12}^{>} + \mathbf{e}_{11}^{>}\,\mathbf{e}_{12}^{<}\,\mathbf{e}_{22}^{<} + \mathbf{e}_{11}^{<}\,\mathbf{e}_{21}^{>}\,\mathbf{e}_{22}^{>} + \mathbf{e}_{11}^{<}\,\mathbf{e}_{21}^{>}\,\mathbf{e}_{22}^{<} + \mathbf{e}_{11}^{>}\,\mathbf{e}_{12}^{<}\,\mathbf{e}_{22}^{>} + \mathbf{e}_{11}^{<}\,\mathbf{e}_{21}^{=} + \\
+ \mathbf{e}_{11}^{>}\,\mathbf{e}_{12}^{=} + \mathbf{e}_{11}^{=}\,\mathbf{e}_{22}^{<} + \mathbf{e}_{11}^{=}\,\mathbf{e}_{22}^{>} + \mathbf{e}_{11}^{>}\,\mathbf{e}_{12}^{<}\,\mathbf{e}_{22}^{=} + \mathbf{e}_{11}^{<}\,\mathbf{e}_{21}^{>}\,\mathbf{e}_{22}^{=} = \mathbf{e}_{11}^{=}(\mathbf{e}_{22}^{=} + \mathbf{e}_{22}^{<} + \mathbf{e}_{22}^{>}) + \\
+ \mathbf{e}_{11}^{>}\,\mathbf{e}_{12}^{<}(\mathbf{e}_{22}^{<} + \mathbf{e}_{22}^{>} + \mathbf{e}_{22}^{=}) + \mathbf{e}_{11}^{<}\,\mathbf{e}_{21}^{>}(\mathbf{e}_{22}^{>} + \mathbf{e}_{22}^{<} + \mathbf{e}_{22}^{=}) + \mathbf{e}_{11}^{<}\,\mathbf{e}_{21}^{<} + \mathbf{e}_{11}^{>}\,\mathbf{e}_{12}^{>} + \mathbf{e}_{11}^{<}\,\mathbf{e}_{21}^{=} + \mathbf{e}_{11}^{>}\,\mathbf{e}_{12}^{=} = \\
= \mathbf{e}_{11}^{=} + \mathbf{e}_{11}^{>}\,\mathbf{e}_{12}^{<} + \mathbf{e}_{11}^{<}\,\mathbf{e}_{21}^{>} + \mathbf{e}_{11}^{<}\,\mathbf{e}_{21}^{<} + \mathbf{e}_{11}^{>}\,\mathbf{e}_{12}^{>} + \mathbf{e}_{11}^{<}\,\mathbf{e}_{21}^{=} + \mathbf{e}_{11}^{>}\,\mathbf{e}_{12}^{=} = \\
= \mathbf{e}_{11}^{=} + \mathbf{e}_{11}^{>}(\mathbf{e}_{12}^{<} + \mathbf{e}_{12}^{>} + \mathbf{e}_{12}^{=}) + \mathbf{e}_{11}^{<}(\mathbf{e}_{21}^{>} + \mathbf{e}_{21}^{<} + \mathbf{e}_{21}^{=}) = \mathbf{e}_{11}^{=} + \mathbf{e}_{11}^{>} + \mathbf{e}_{11}^{<} = 1. \blacksquare
\end{gathered}$$